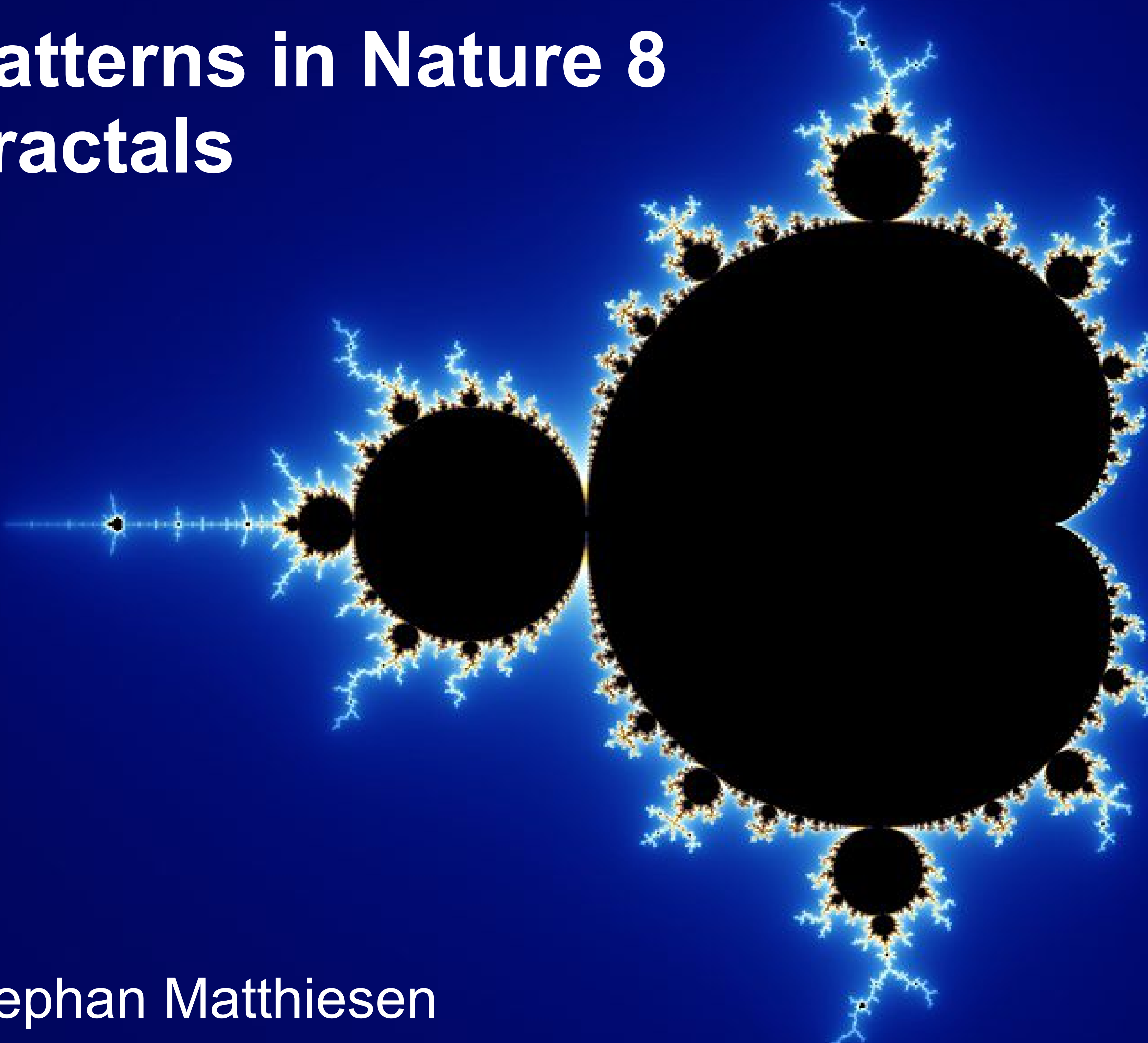
Patterns in Nature 8
Fractals



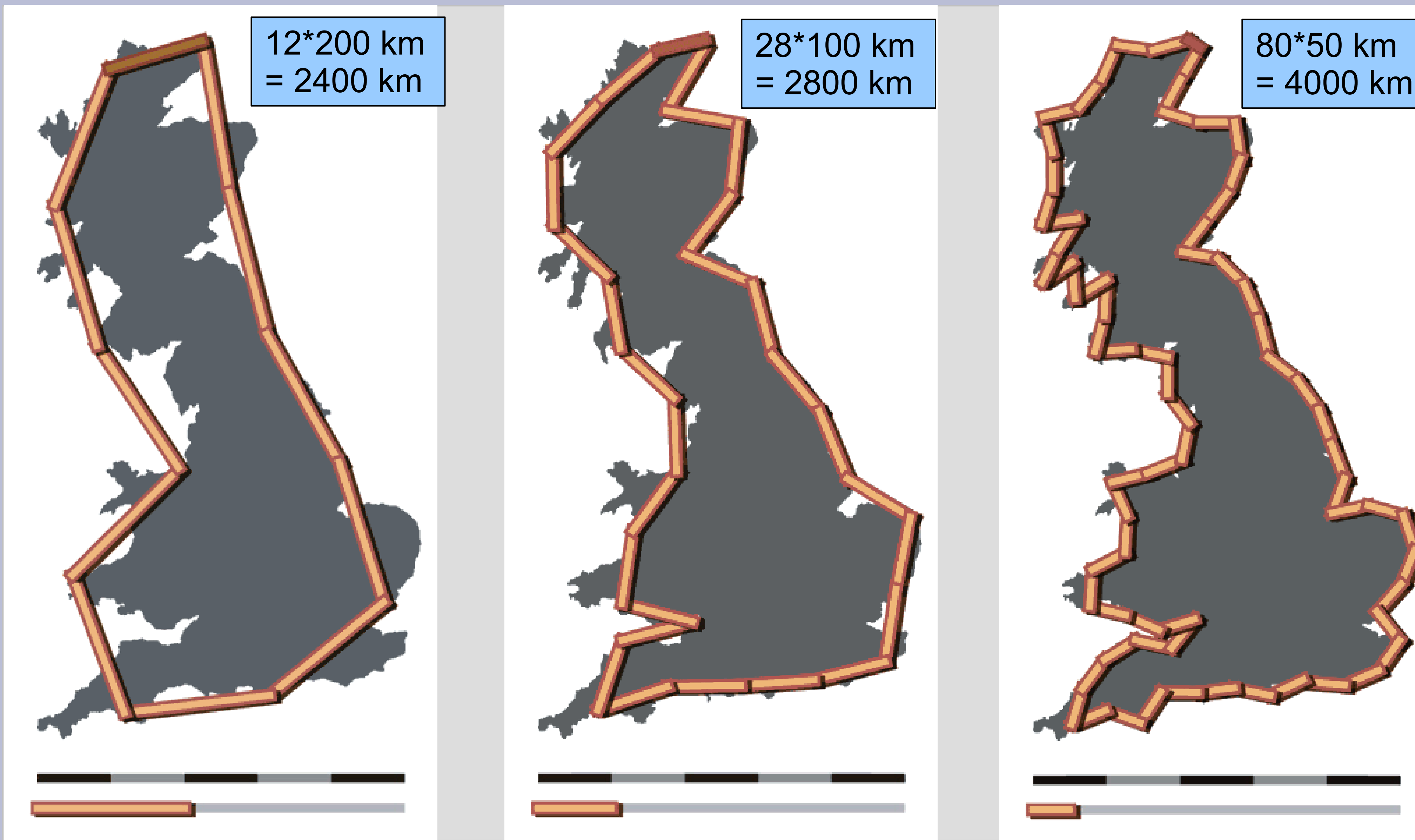
Stephan Matthiesen

How long is the Coast of Britain?

- CIA Factbook (2005): 12,429 km

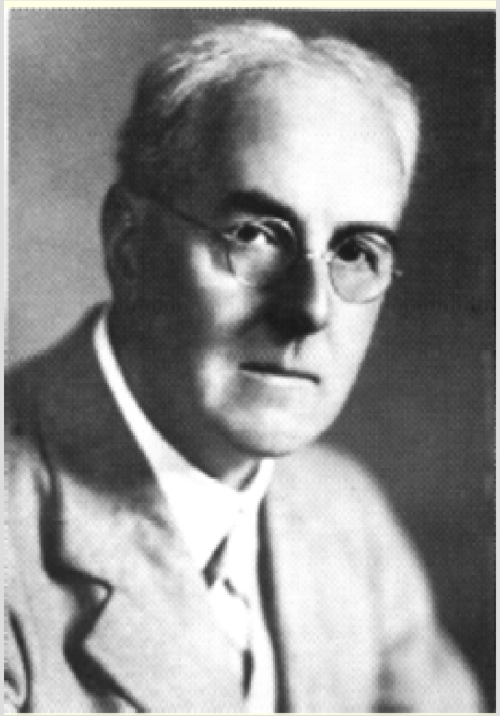
http://en.wikipedia.org/wiki/Lewis_Fry_Richardson

How long is the Coast of Britain?



How long is the Coast of Britain?

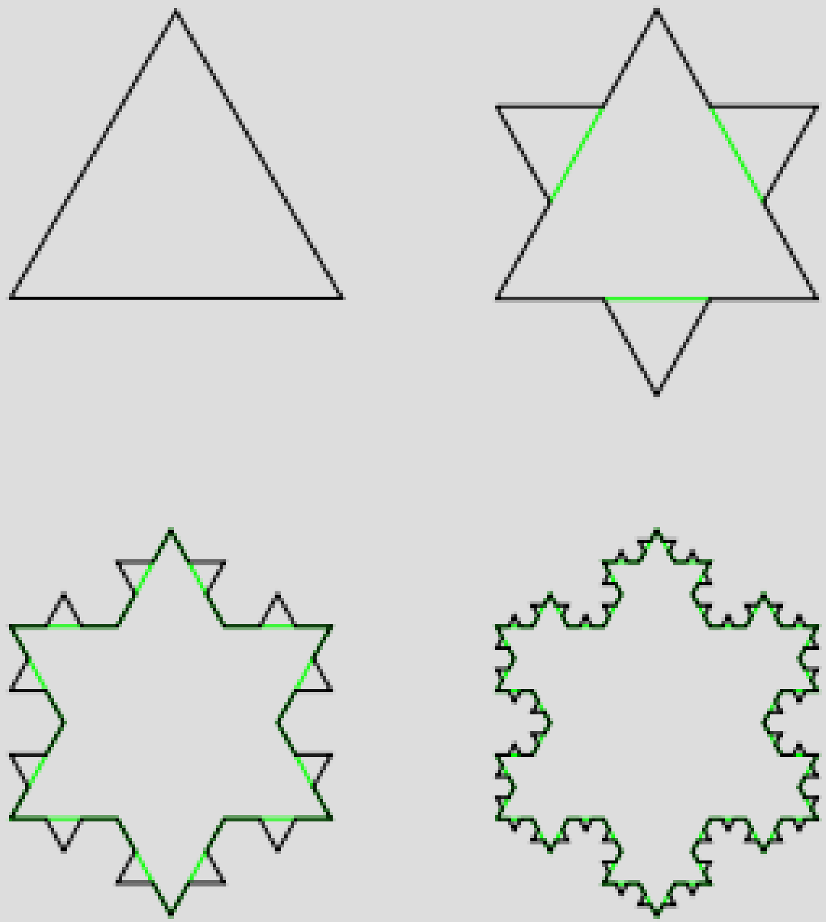
- Discussed by **Lewis Fry Richardson** (1881-1953)
pioneer of numerical weather prediction



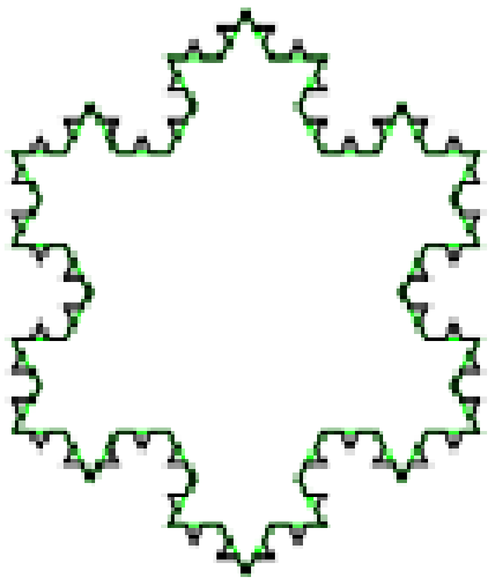
Richardson's „forecast factory“

Simple Example: Koch snowflake

- First introduced by Helge von Koch (1904)
- Defined by an iteration rule:
  - Replace the middle line segment by two sides of an equilateral triangle
  - repeat infinitely...
- Hausdorff dim.: log(4)/log(3)=1.26



Frequent features of fractals



- F is **self-similar** (at least approximately or stochastically).
- F has a **fine-structure**: it contains detail at arbitrary small scales.
- F has a simple definition.
- F is obtained through a **recursive** procedure.
- The geometry of F is not easily described in classical terms.
- It is awkward to describe the local geometry of F.
- The size of F is not quantified by the usual measures of length (this leads to the **Hausdorff dimension**)

(after Falconer 1990)

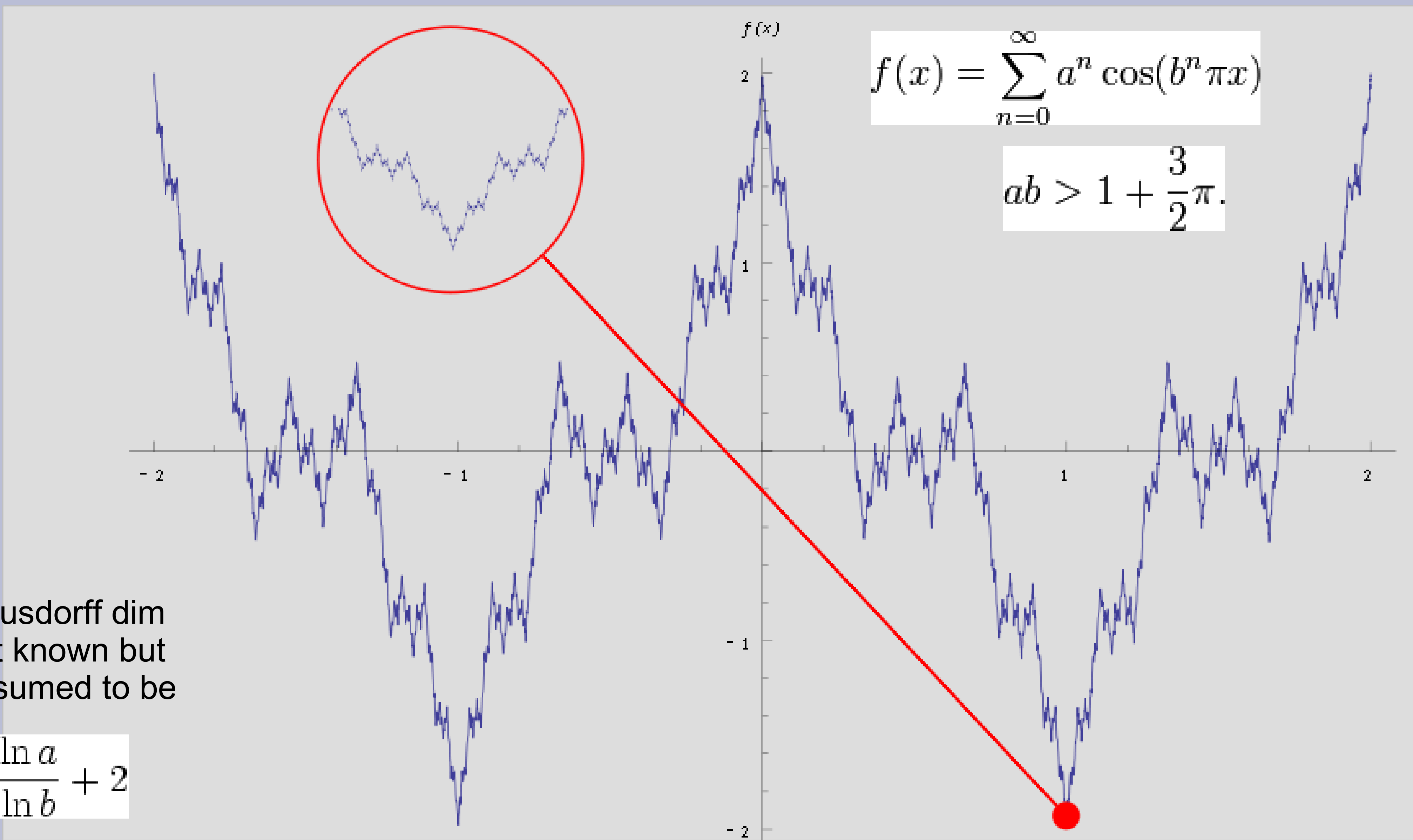
Fractals

- Popularized (1967, Nature) by **Benoît B. Mandelbrot** (1924-2010) he invented the term fractal

- But: **Karl Weierstraß** (1815-1897) defined continuous non-differentiable functions

Weierstraß function (18 Jul 1872)

a pathological example of a real-valued function everywhere continuous but nowhere differentiable



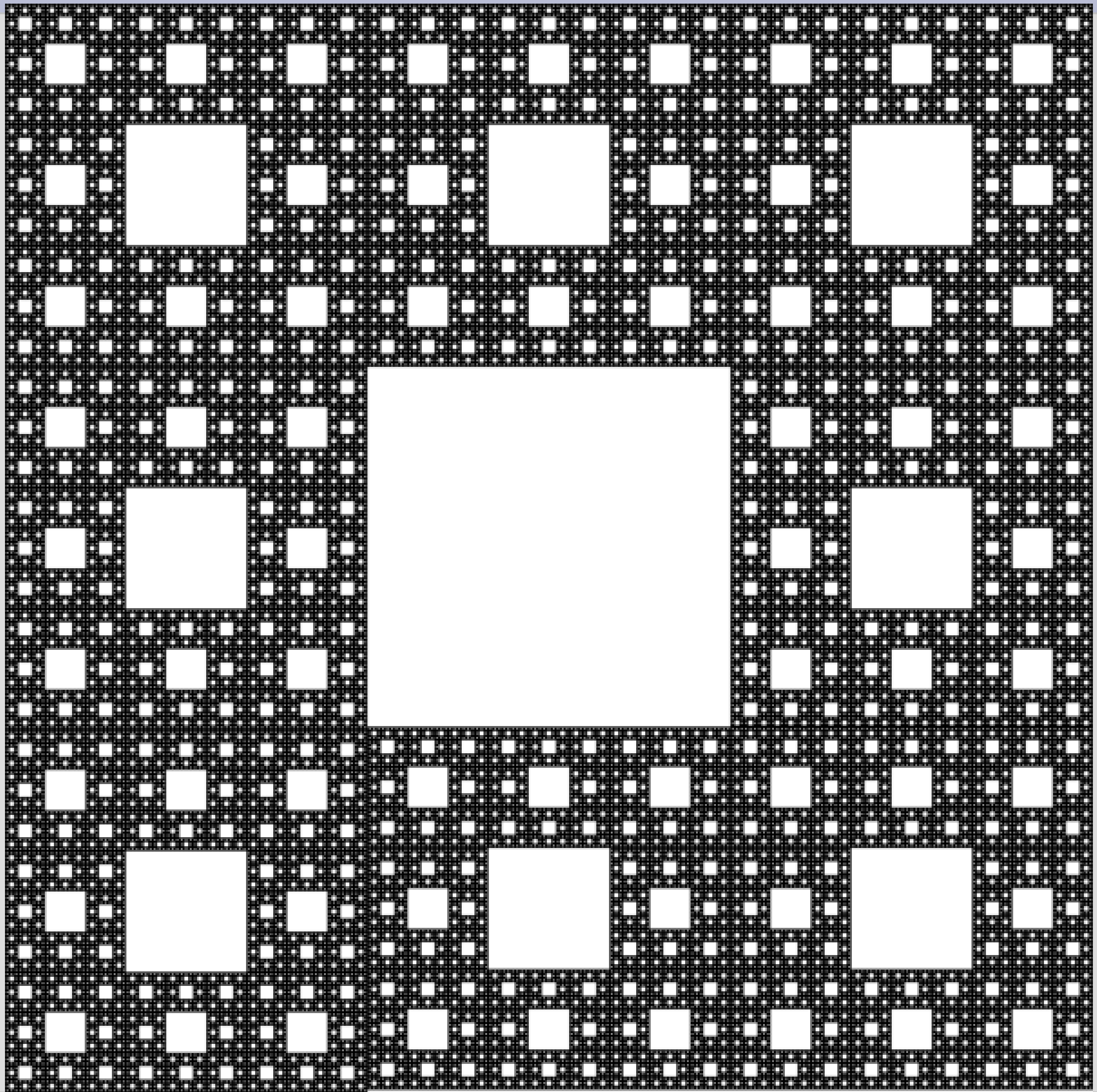
$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

$$ab > 1 + \frac{3}{2}\pi.$$

Hausdorff dim not known but assumed to be

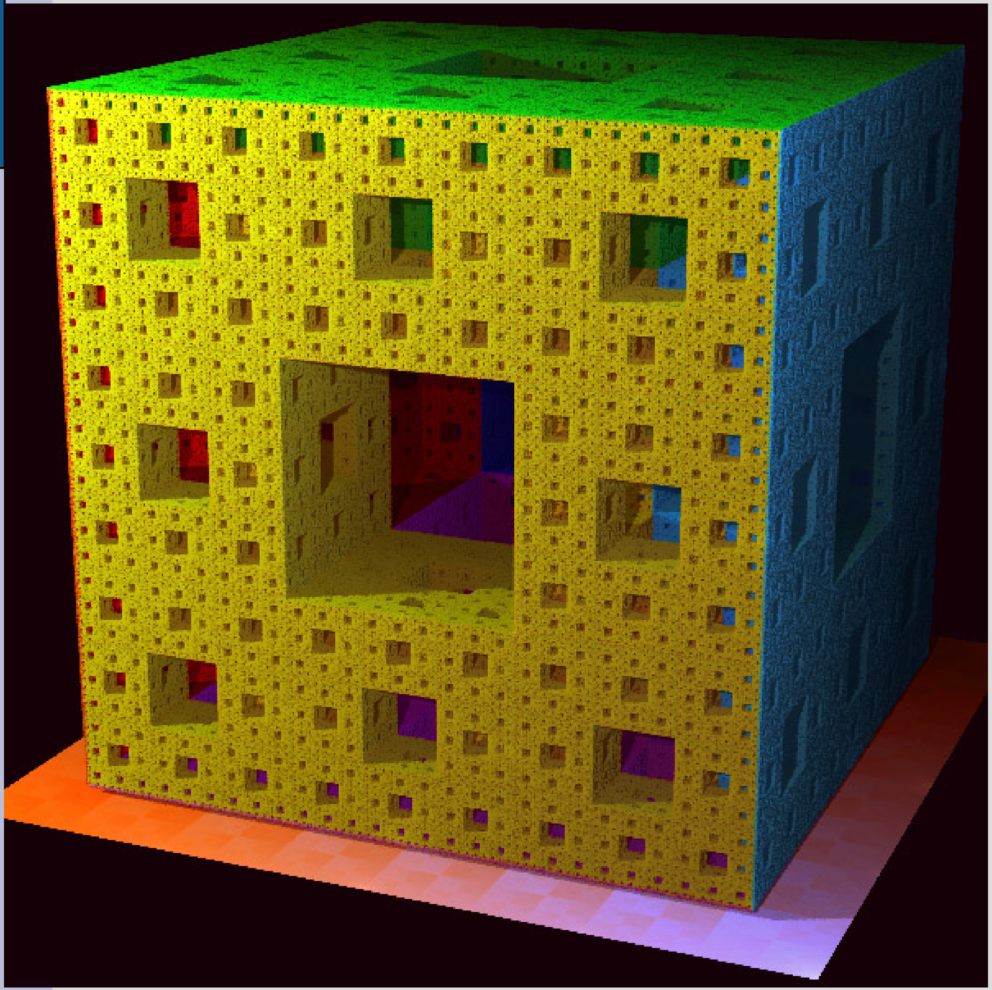
$$\frac{\ln a}{\ln b} + 2$$

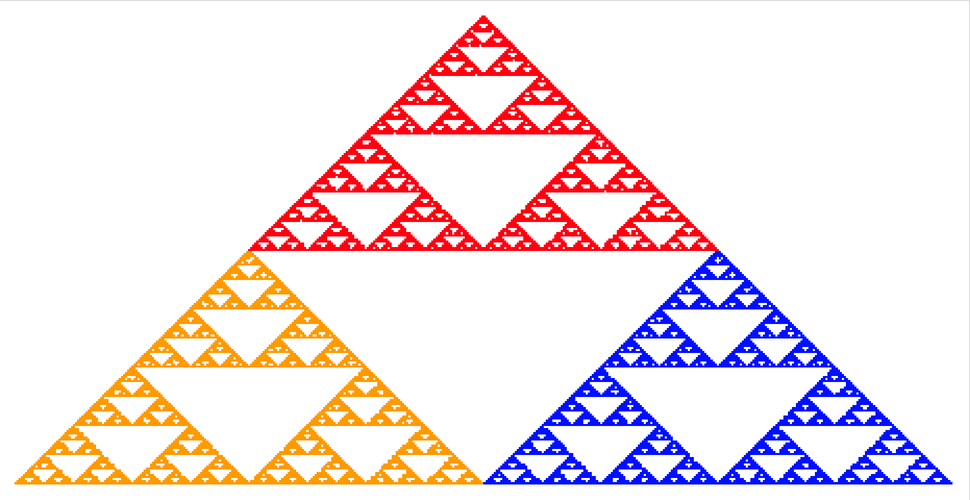
Sierpinski carpet and triangle

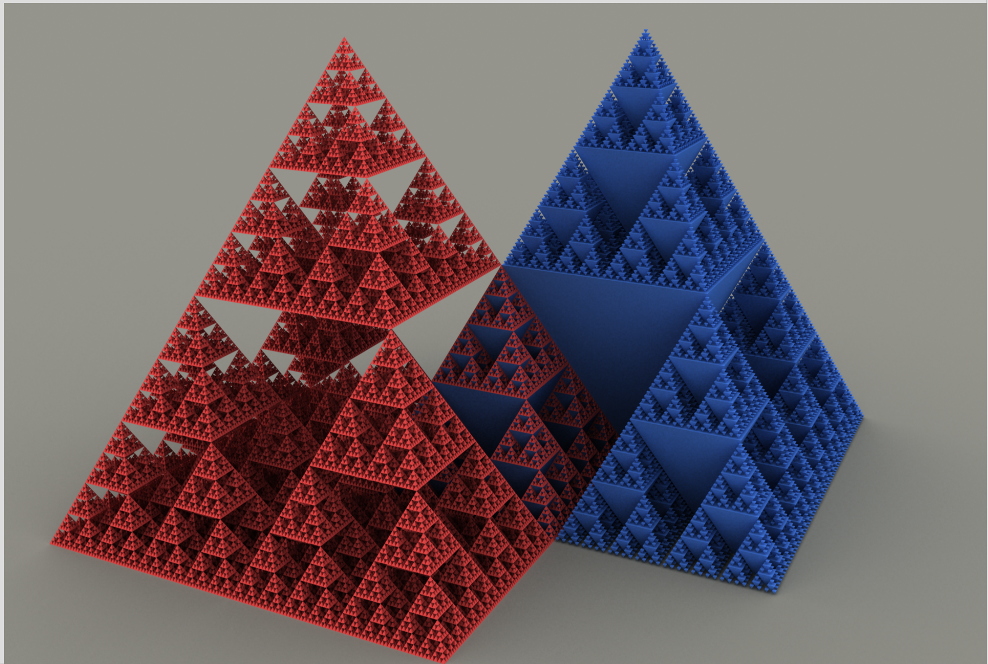


$$D = \log 8 / \log 3 = 1.8928$$

Menger sponge and Sierpinski pyramid







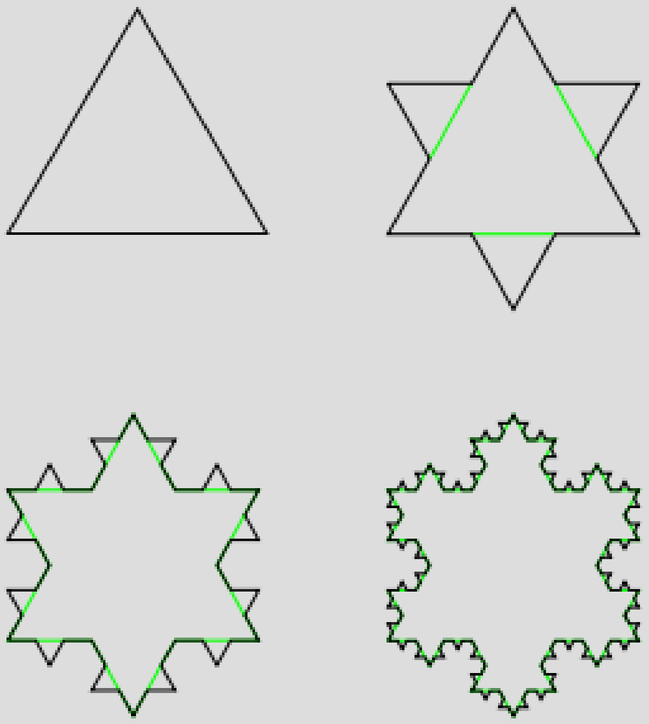
Dimension

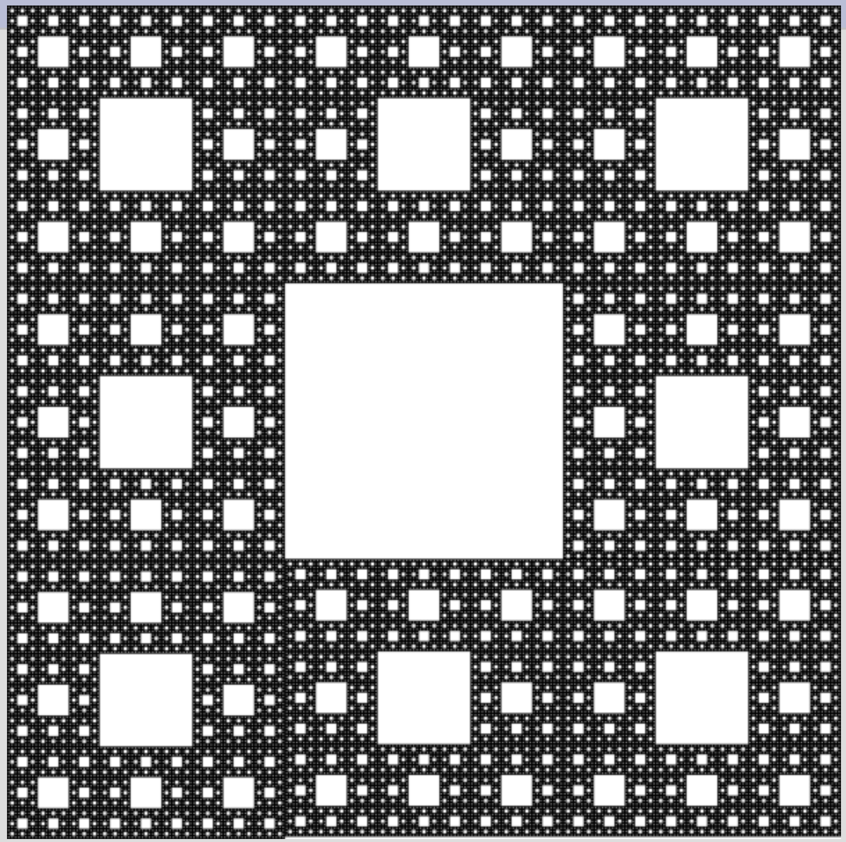
- “Normal” dimension (affine dimension of a vector space) defined by number of coordinates that are needed to define a point
- Covering dimension (“topological” dimension) make the cover half as large (a=2)
  - 1-d: make line segments half as long => you need N = 2 = 2^1 times as many (N=2).
  - 2-d: make squares half as large => you need N = 4 = 2^2 times as many.
  - 3-d: make cubes half as large => you need N = 8 = 2^3 times as many.
- Extend this to fractional numbers

$$a^D=N$$

$$D=\log(N)/\log(a)$$

Hausdorff dimension





$3^D = 4$

$3^D = 8$

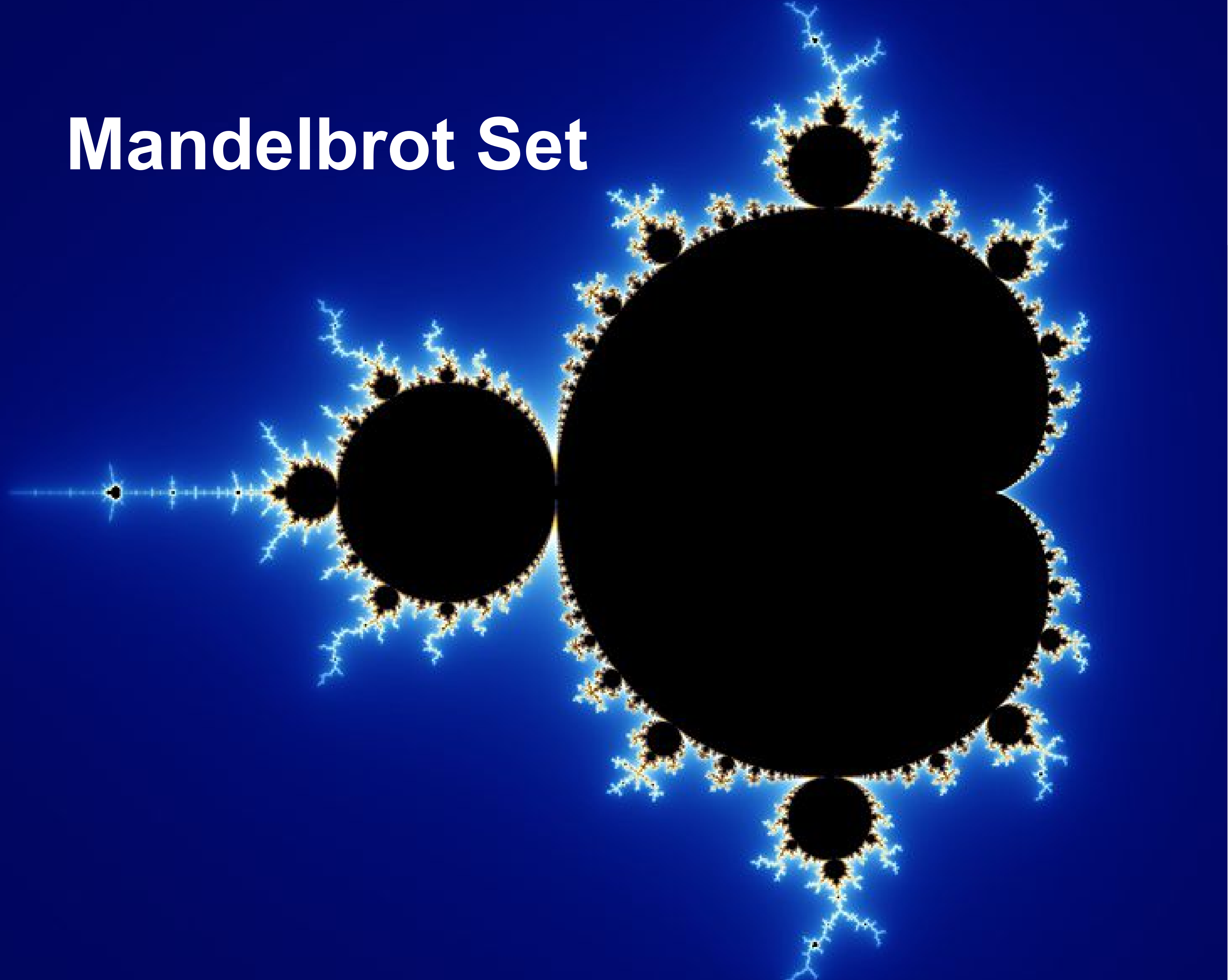
D = log4/log3 = 1.26

D = log8/log3 = 1.8928

Hausdorff dimension

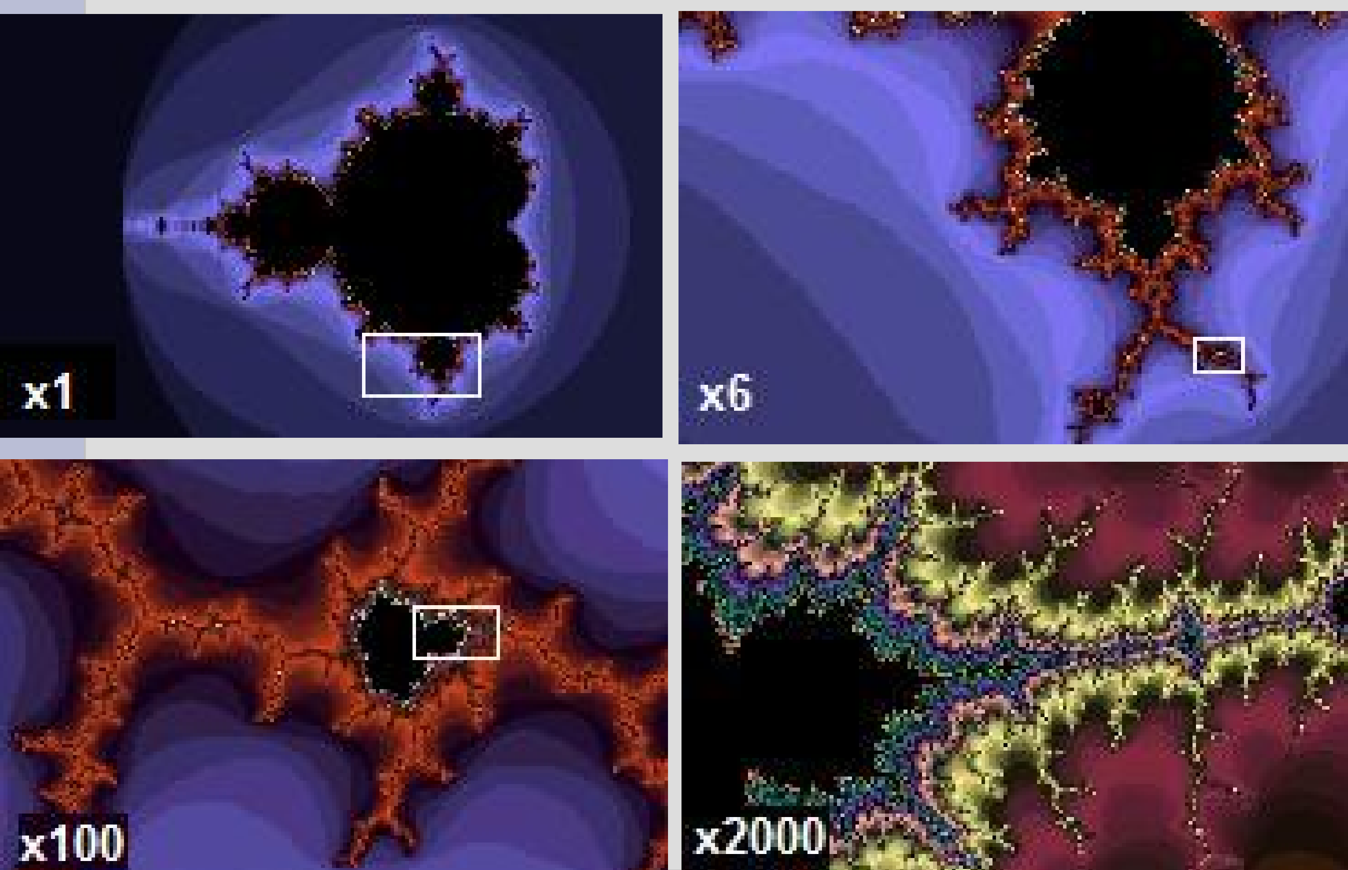
- Defined 1918 by Felix Hausdorff
- Also called
  - Hausdorff-Besicovitch dimension
  - Fractal dimension
  - Capacity dimension
- Can be any real number (unlike the “normal” [integer!] dimension)

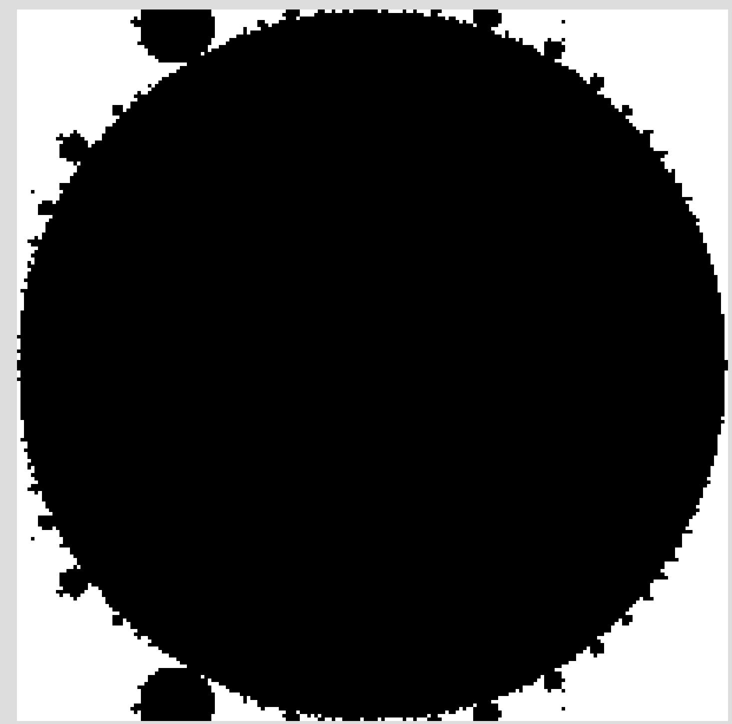
Mandelbrot Set



Mandelbrot Set

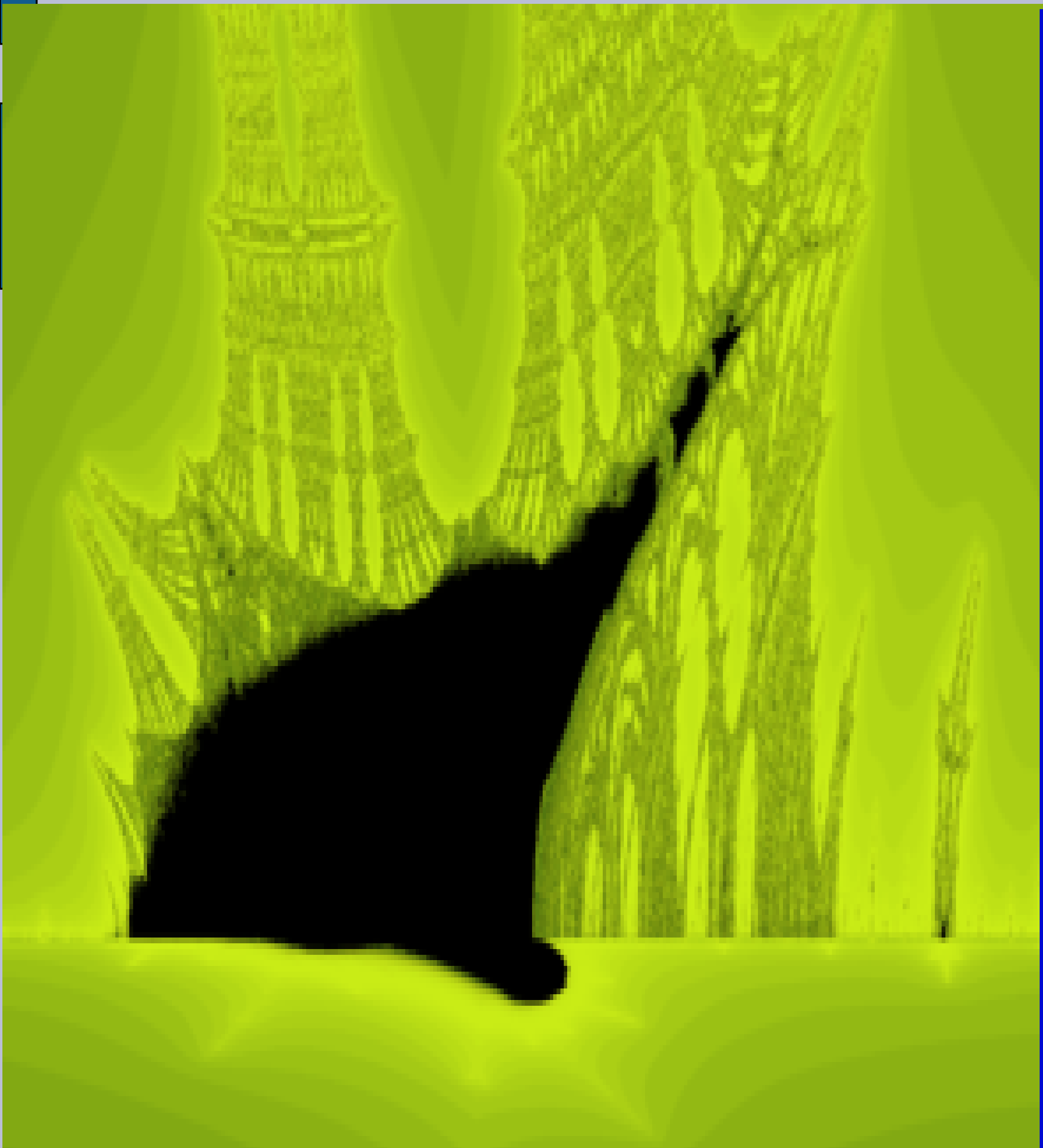
- Definition:
  - for every c calculate the iteration $P_c : z \mapsto z^2 + c$
  - if result remains finite, then c belongs to the set

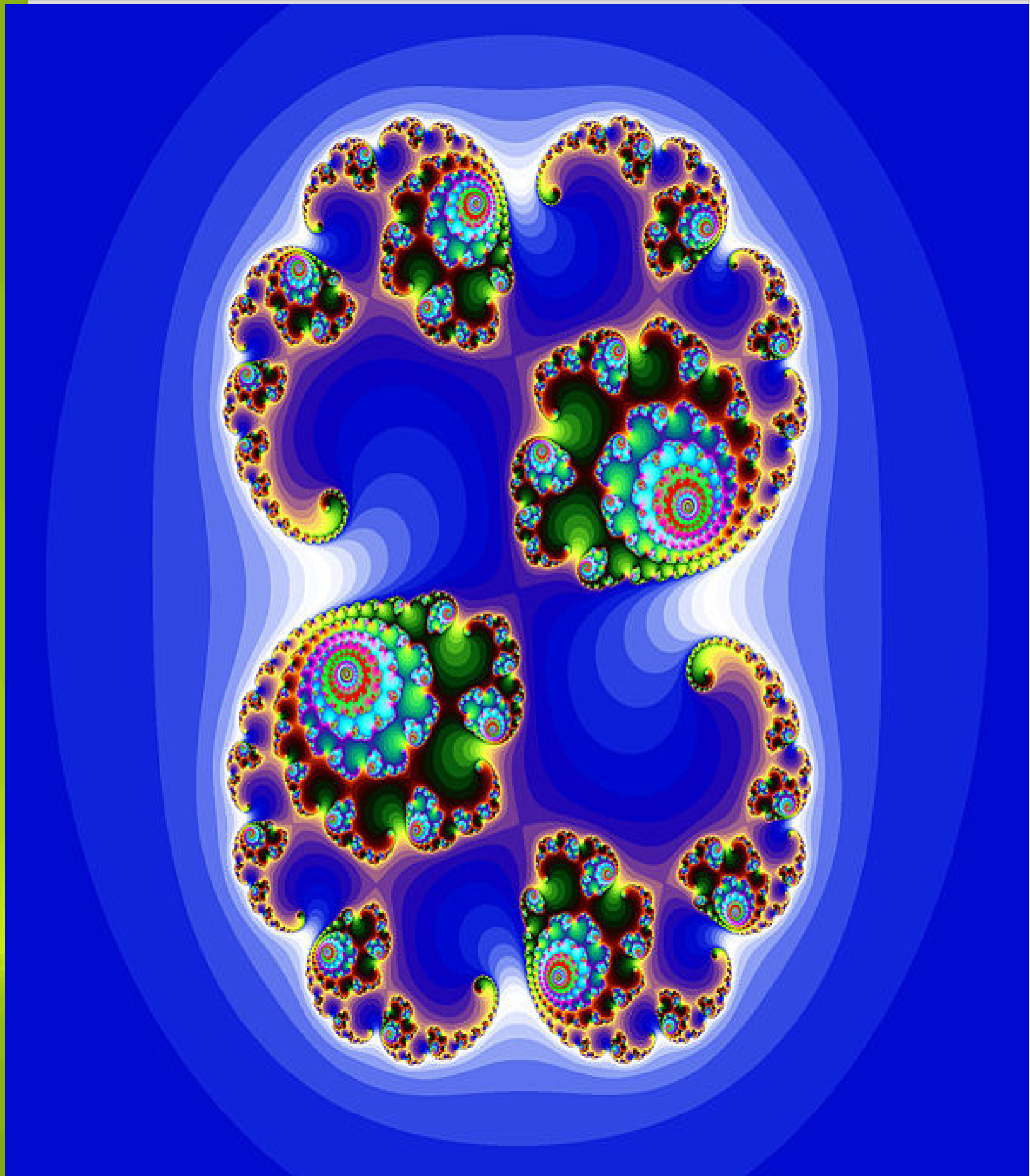




Self-similarity

More fractals: “burning ship” and Julia set



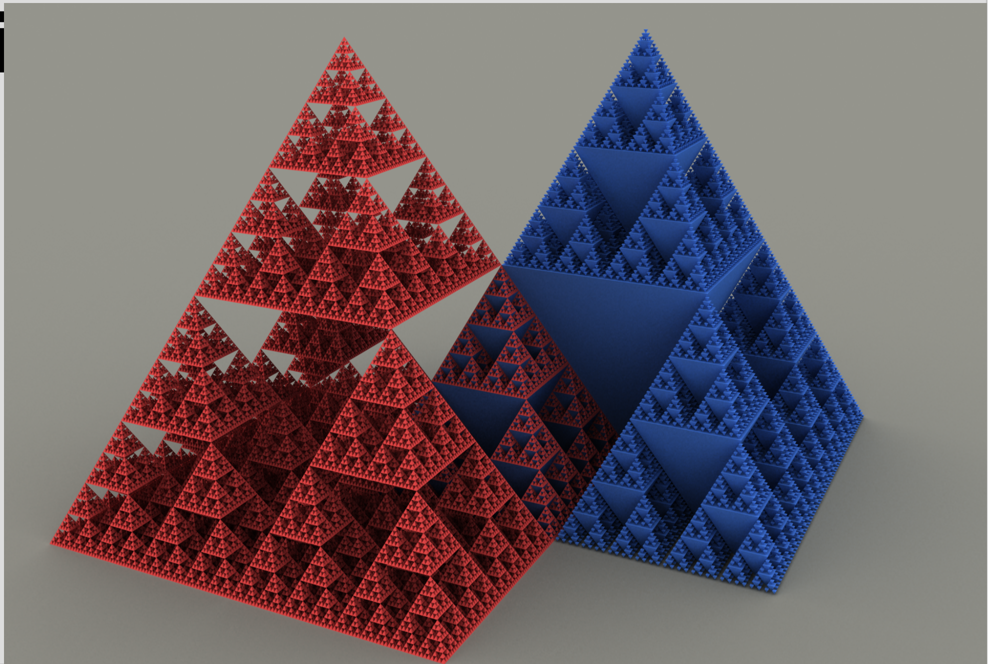


Generation of fractals in mathematics

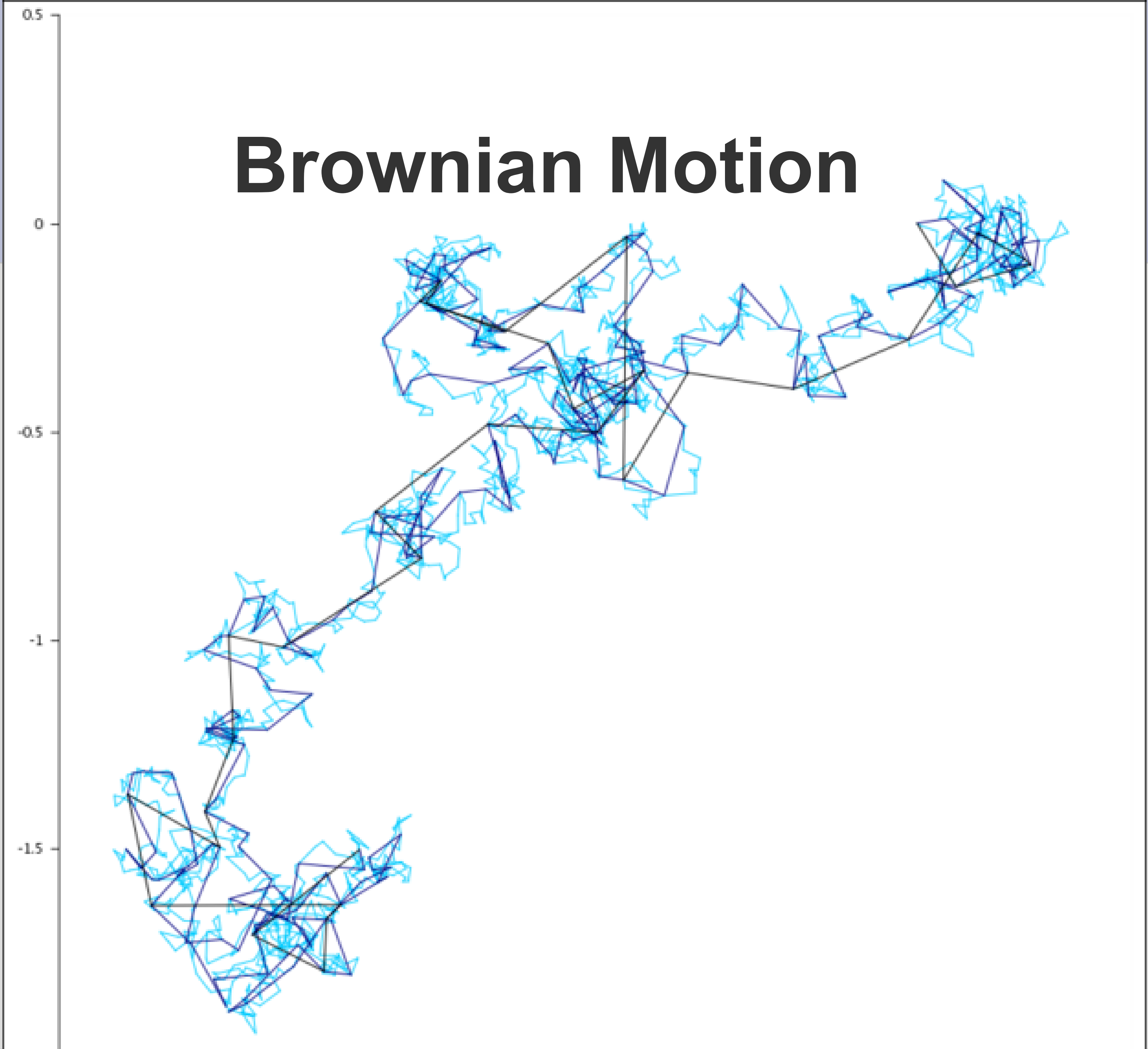
- **Escape time fractals**:
Recurrence relation at each point in space
e.g. Mandelbrot set, Julia set
- **Iterated function fractals**:
Fixed geometric replacement rule
e.g. Koch snowflake,
- **Random fractals**:
Stochastic (not deterministic) processes
e.g. random walk (Brownian motion), fractal landscapes

Self-similarity

- **Exact self-similarity**:
  Fractal appears identical at different scales
- **Quasi-self-similarity**:
  Fractal appears approximately identical at different scales
- **Statistical self-similari**
  Some numerical or statistical measure is preserved across scales

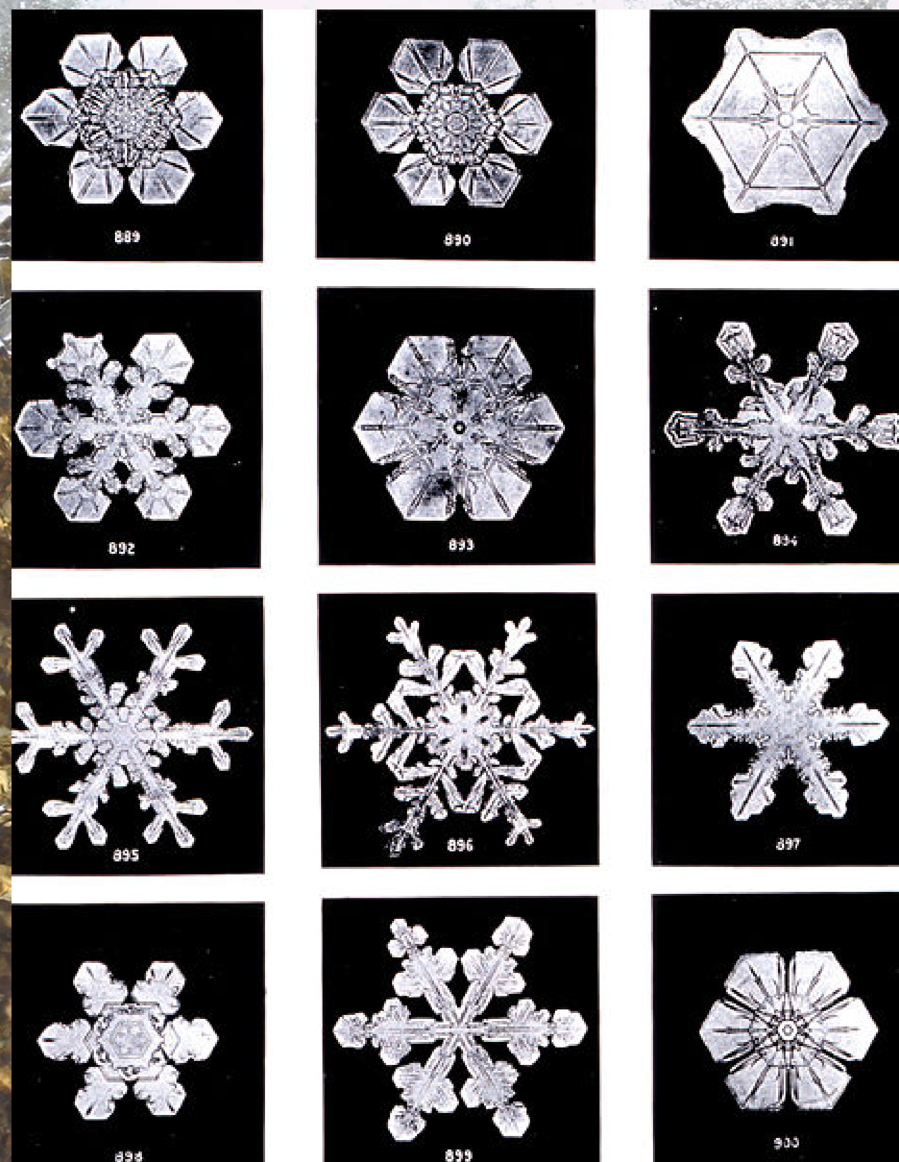


Brownian Motion

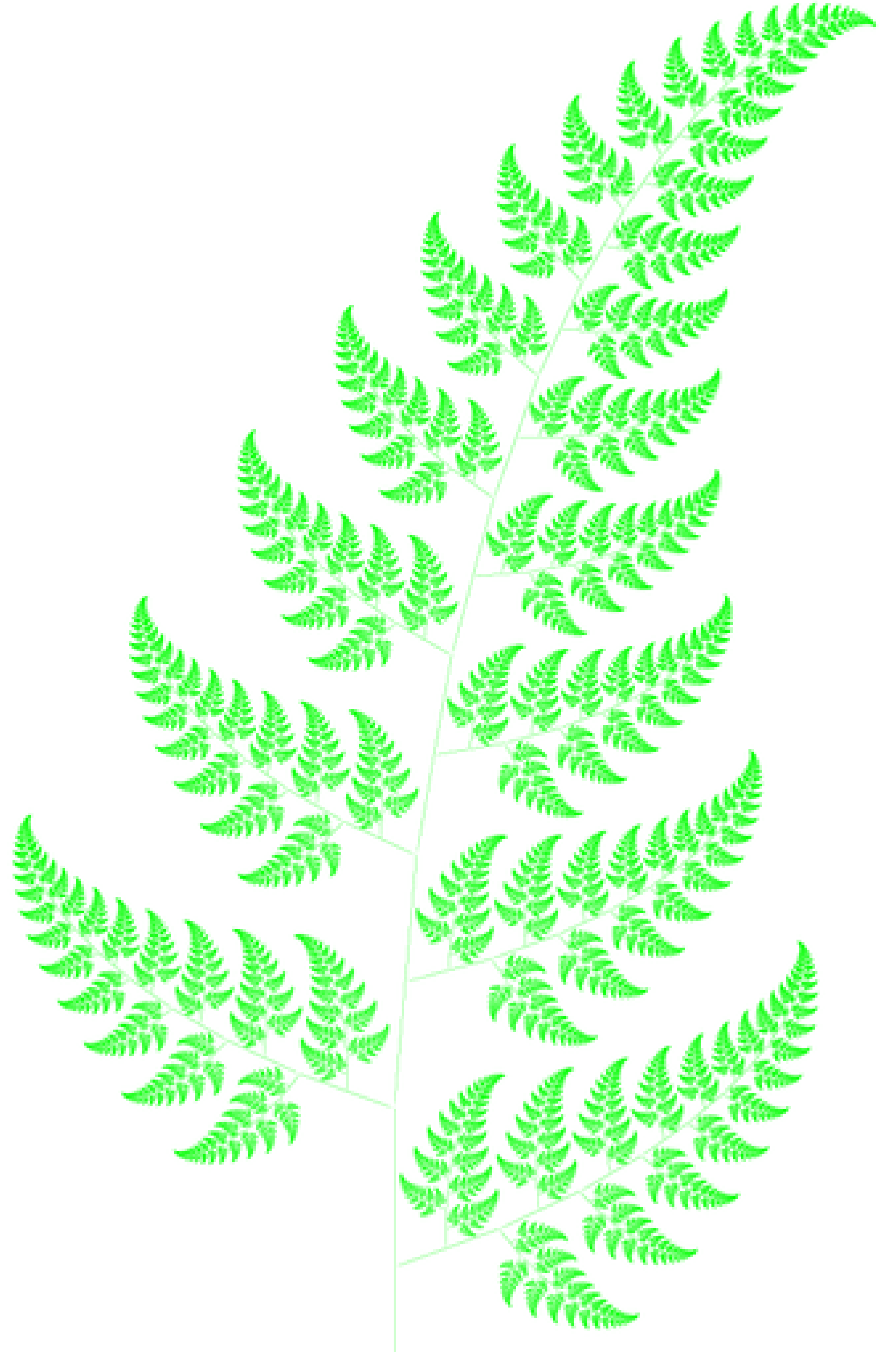


Snowflakes

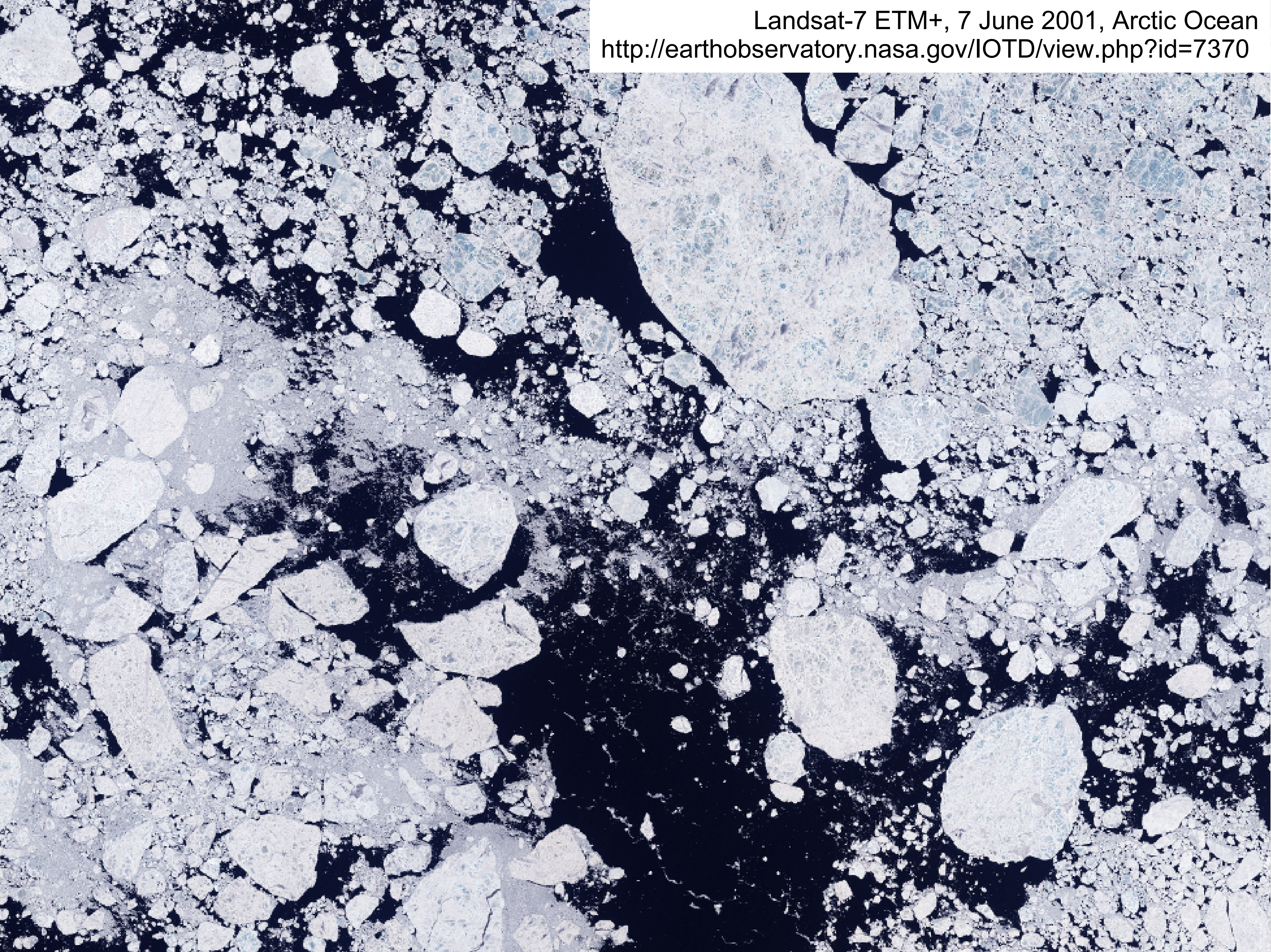
Wilson Bentley (1865-1931)




Manganese oxide dendrites

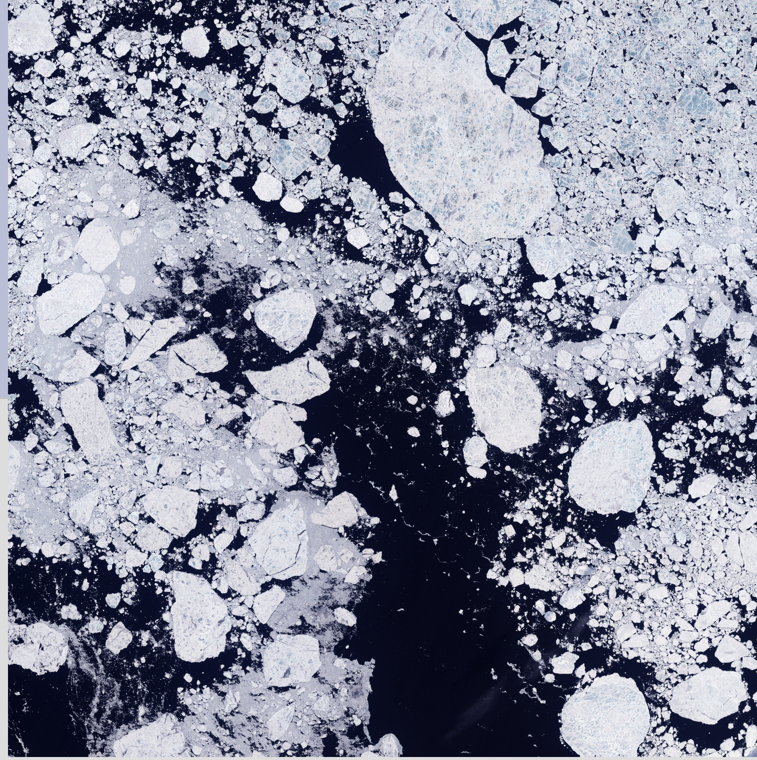


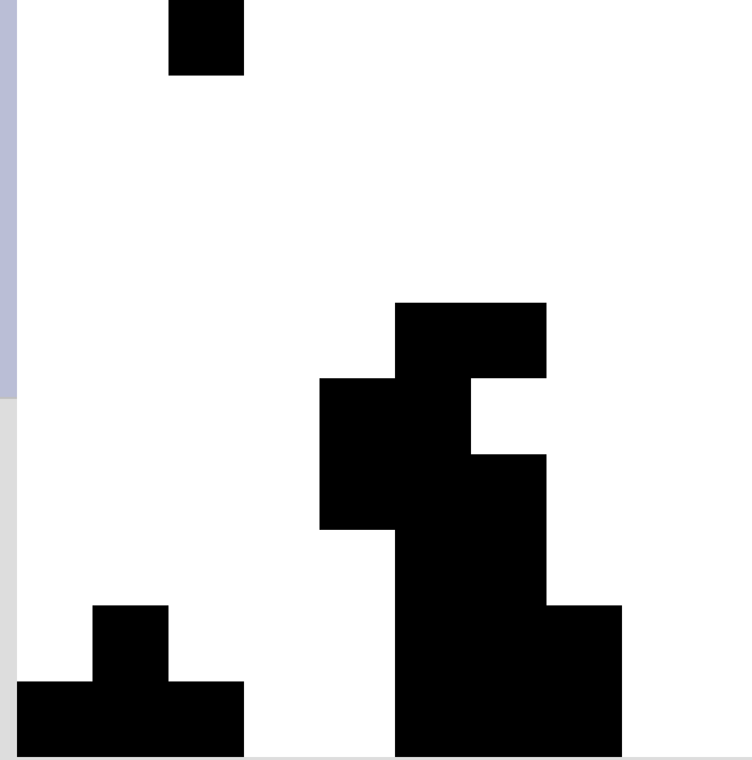
Bransley's fern
(computer generated)


Landsat-7 ETM+, 7 June 2001, Arctic Ocean
http://earthobservatory.nasa.gov/IOTD/view.php?id=7370

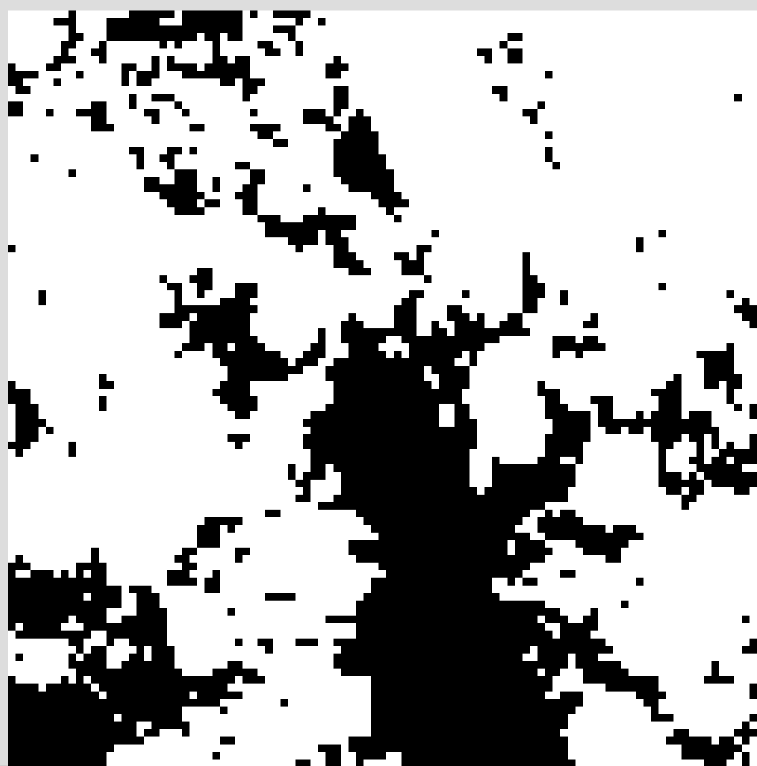
Full image:
6,975,486
/ 10,614,564

= **65.7%** ice





80 ice pixels out of 100 = **80%** ice cover



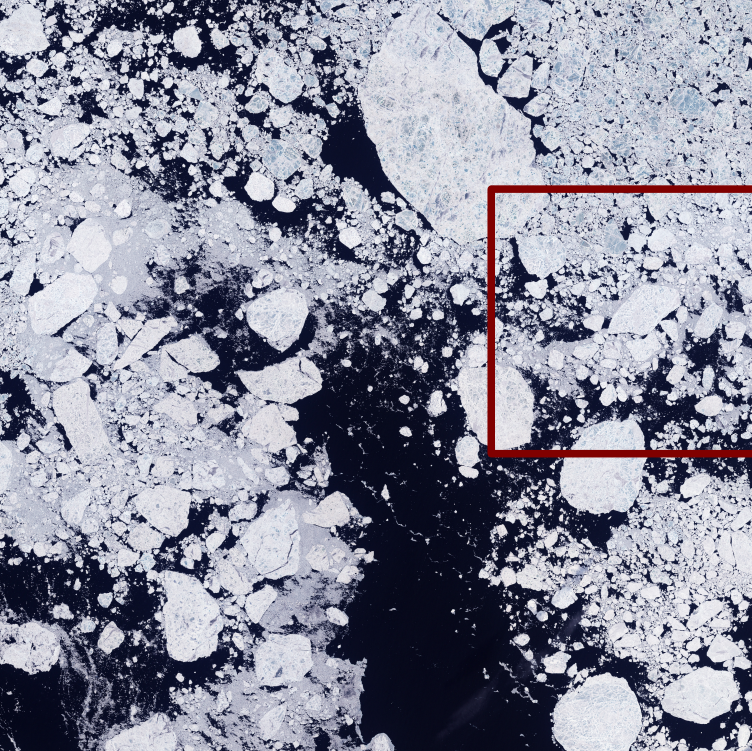
Dimension (using the 10x10 and 100x100 images):

a = 10
N = 7010/80 = 87.625

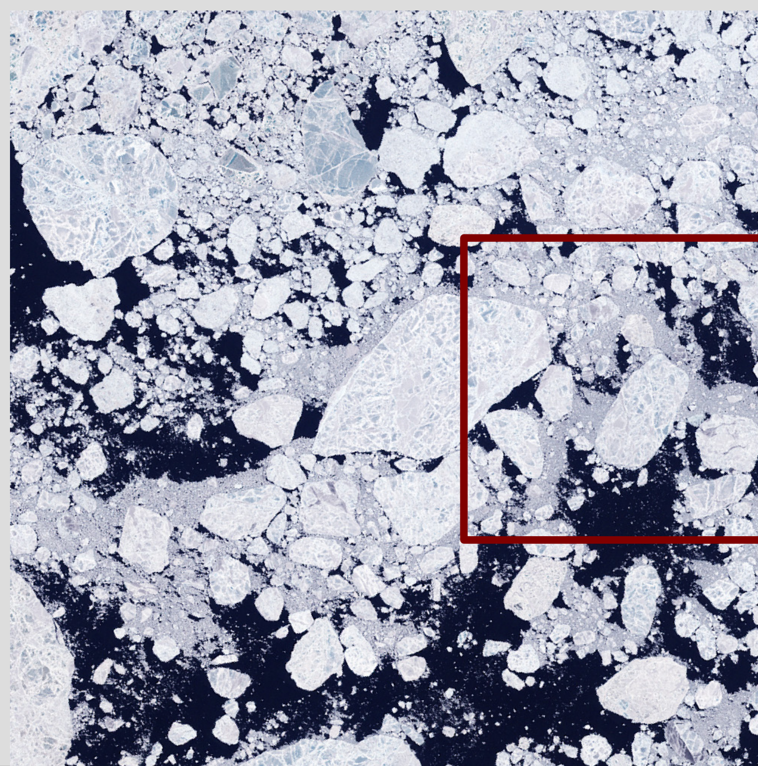
dim = log (N) / log (a) = 1.94

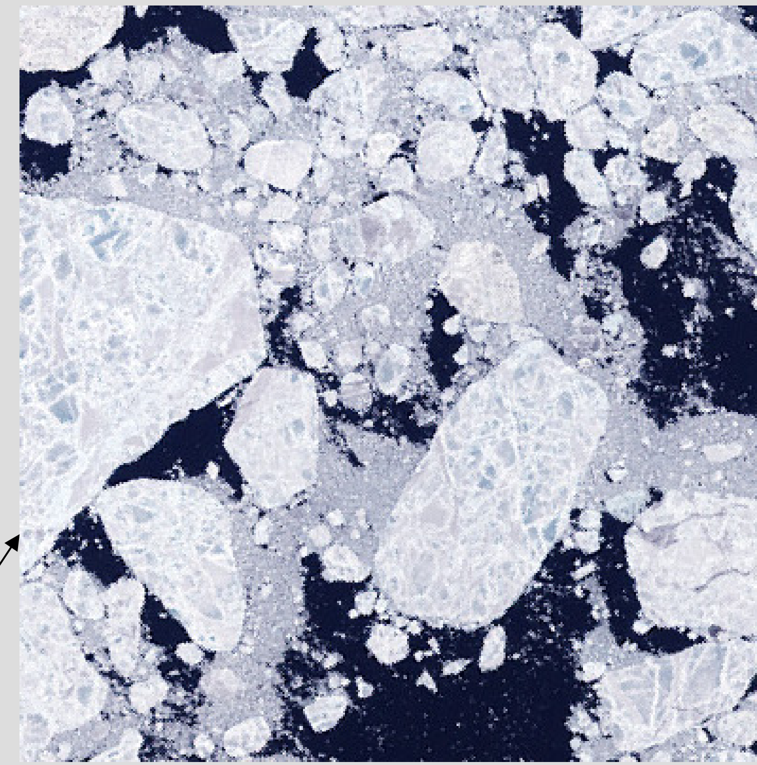
7010 ice pixels / 10000 in total
= **70.1%** ice cover

Self similarity

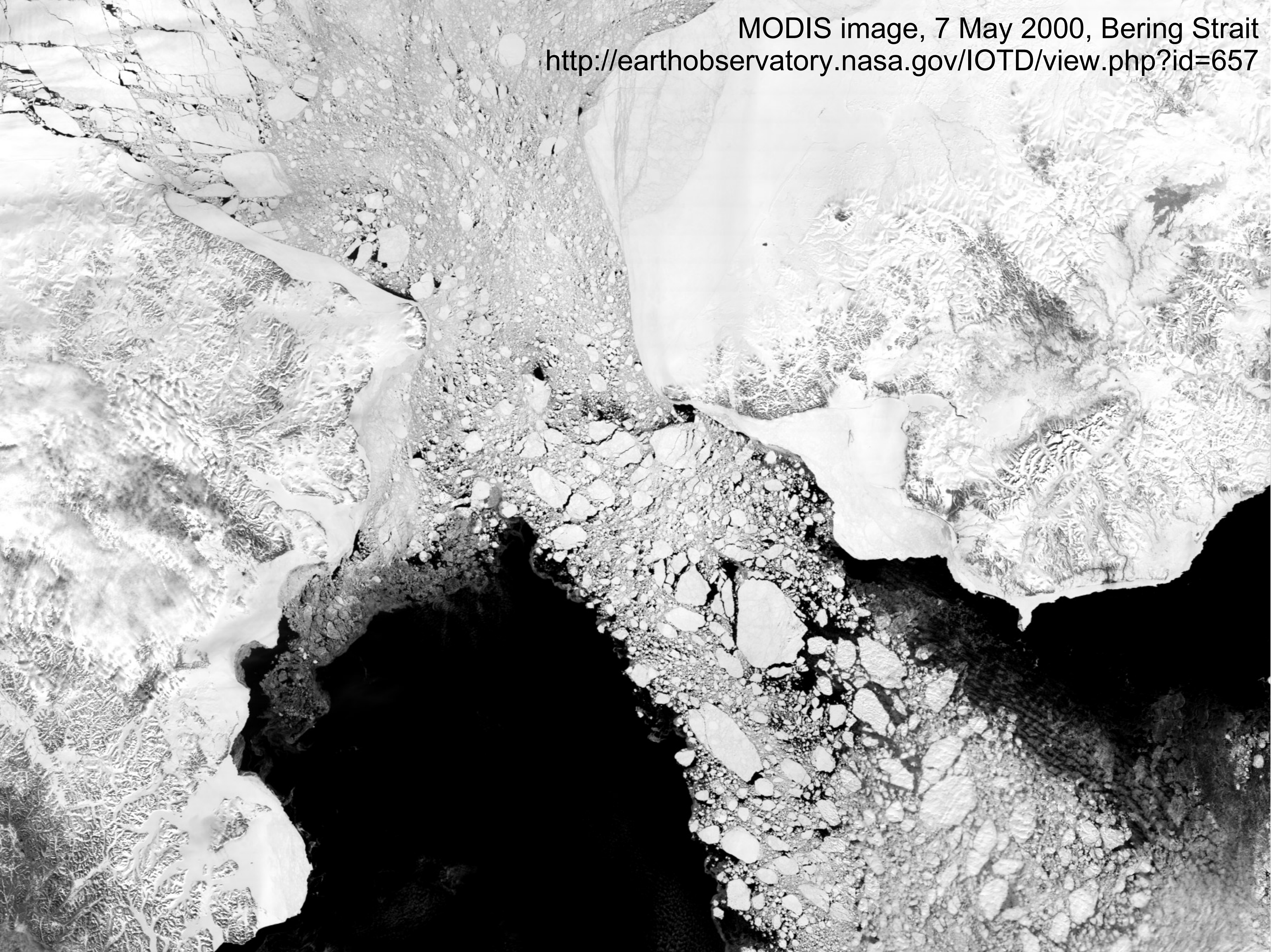


10km

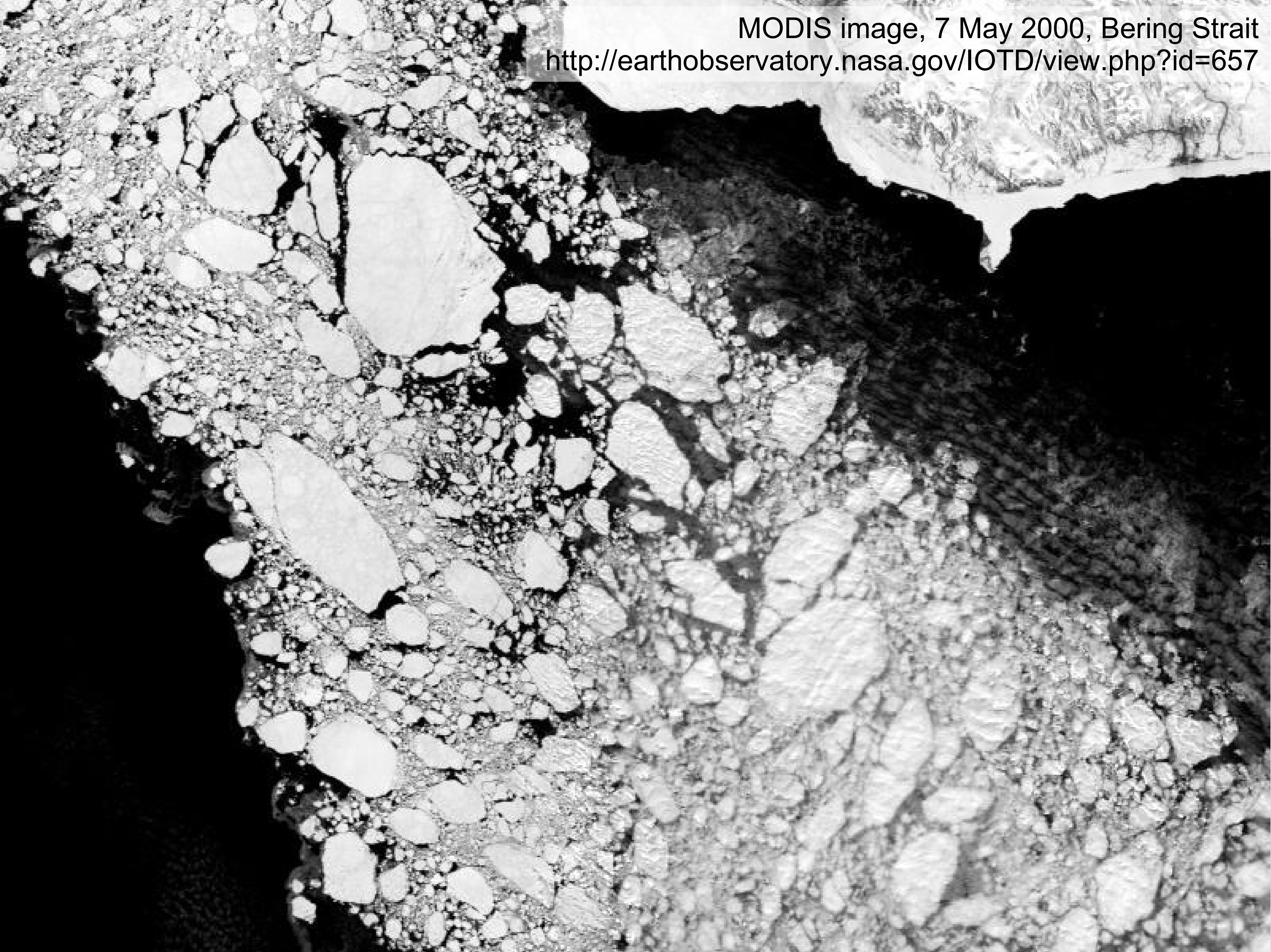




1km



MODIS image, 7 May 2000, Bering Strait
http://earthobservatory.nasa.gov/IOTD/view.php?id=657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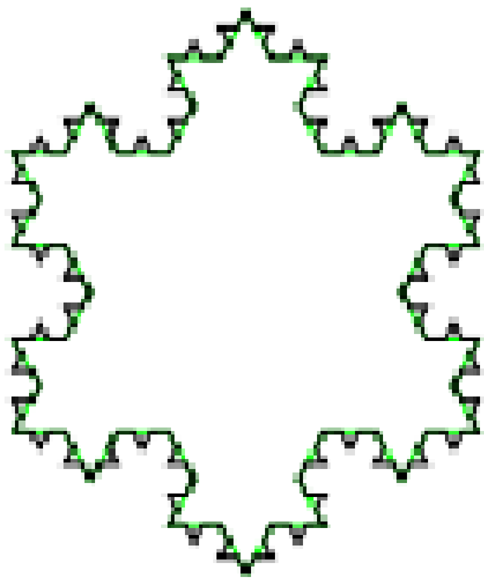
Frequent features of fractals



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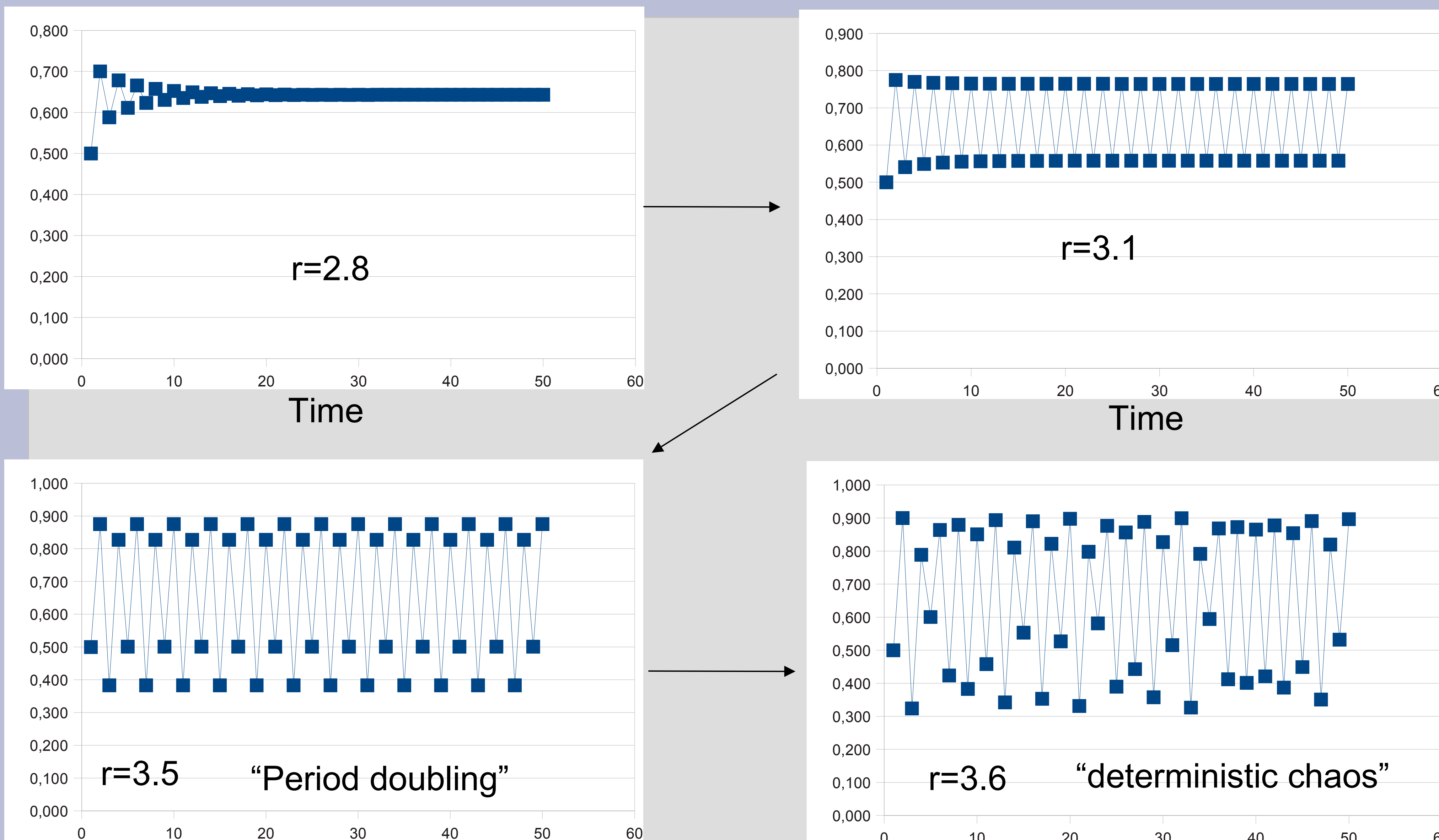
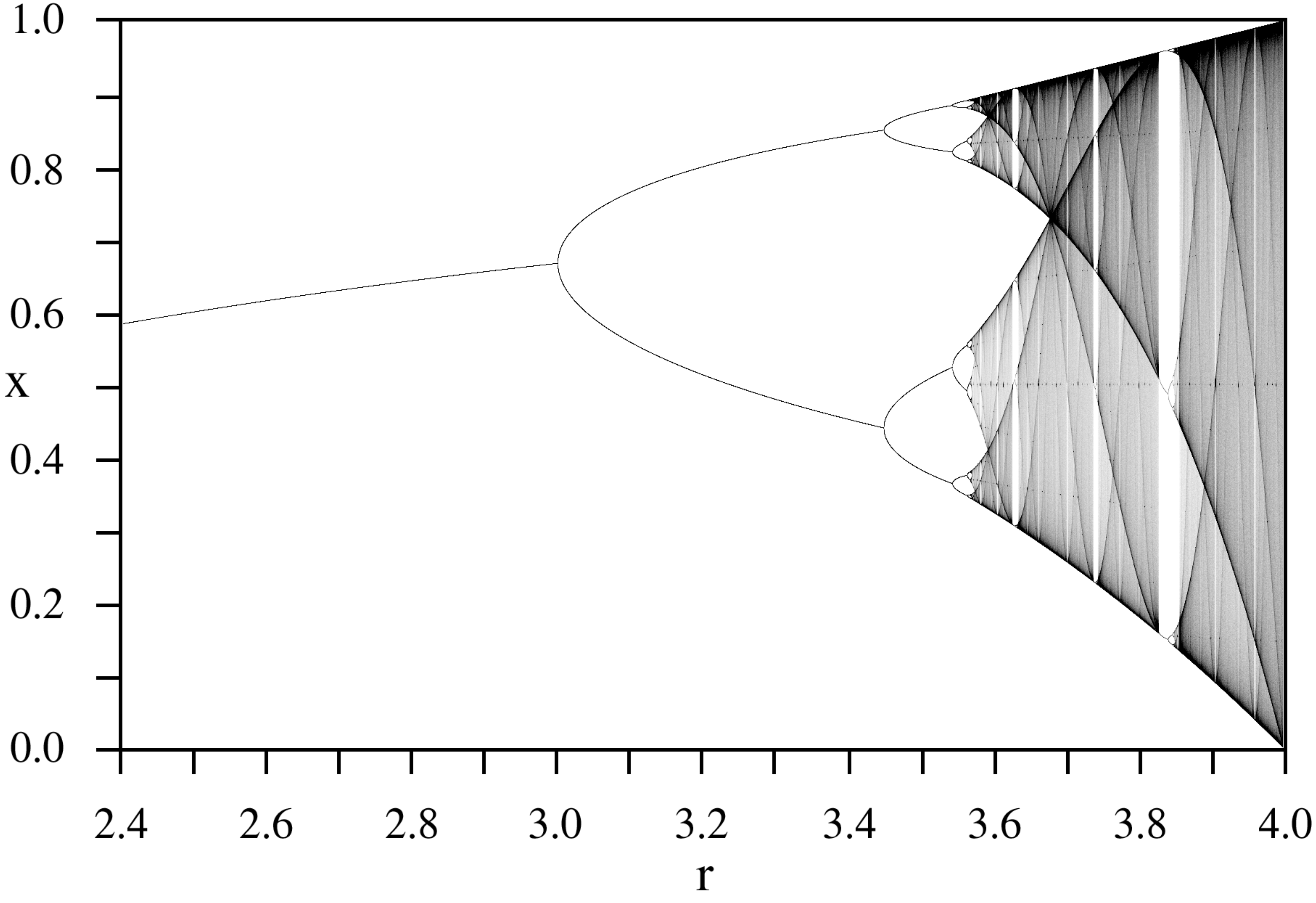
(after Falconer 1990)

The link to Chaos I:
The logistic map

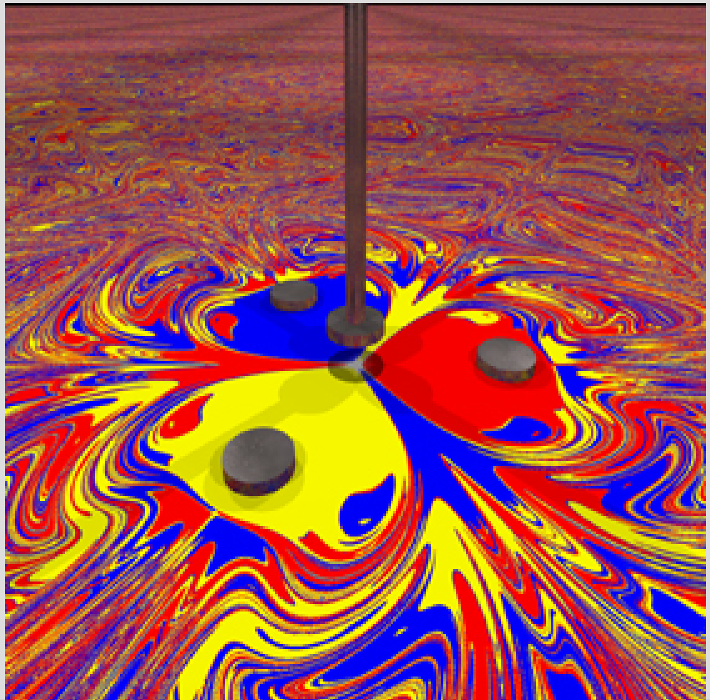


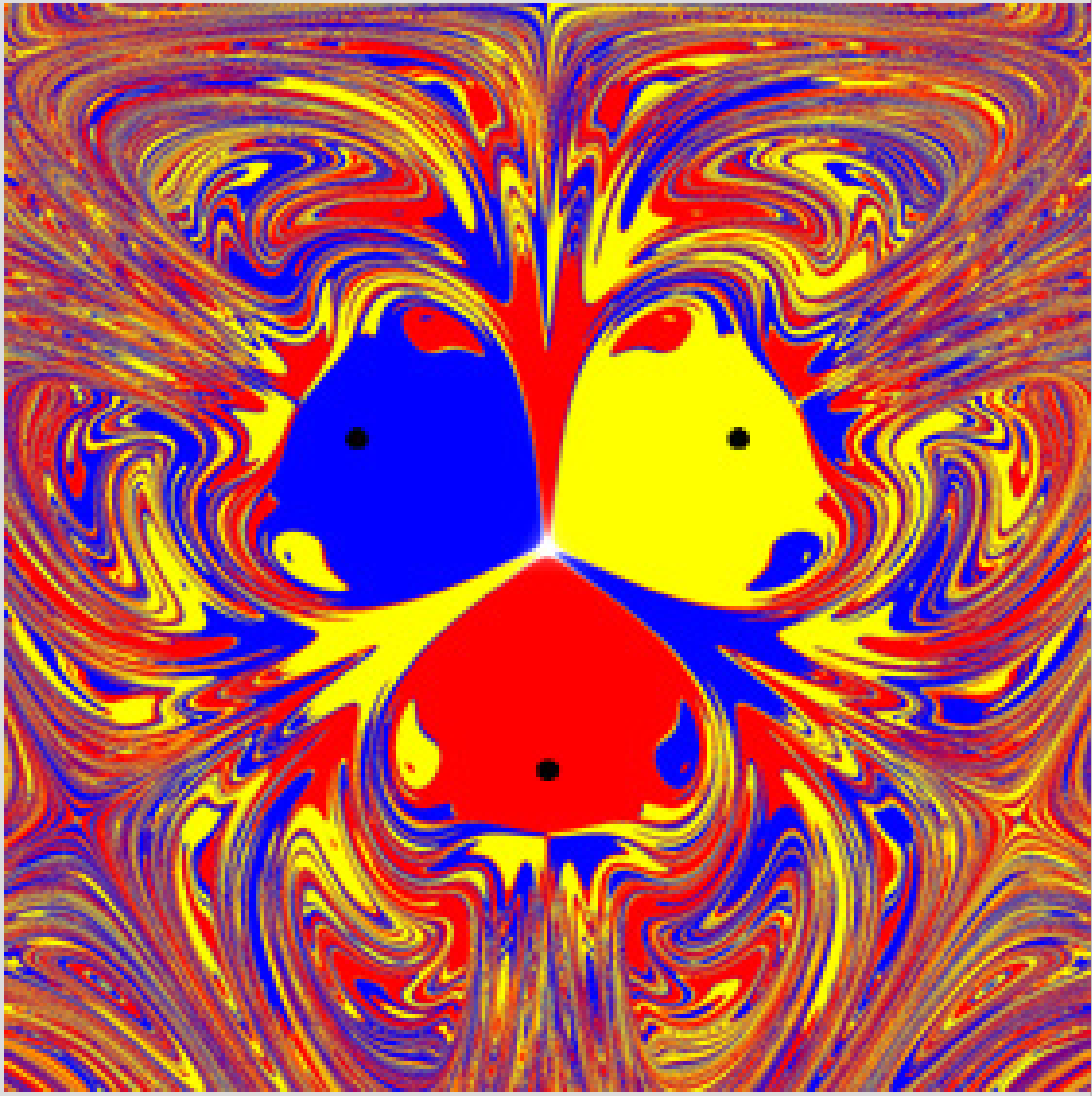
Bifurcation diagram



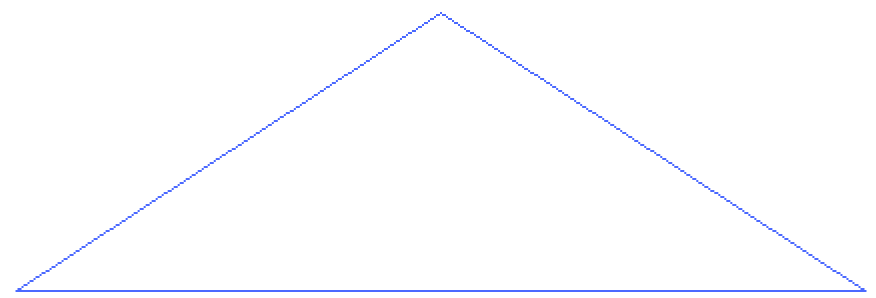
The link to Chaos II: The magnetic pendulum

For each starting point, calculate the final resting position (over one of the three magnets)
Then colour the starting point accordingly
Result: a fractal!





Fractal landscape



Random midpoint displacement

General remarks

- Fractal geometry describes shapes
- It does not explain mechanisms how these shapes grow (although it can provide some constraints for possible mechanisms)
- It not inform us about the (evolutionary) function of an (biological) objects

Patterns in Nature Outline

1. Introduction
2. Waves and oscillations
3. Regularity and chaos
4. Animal cooperation
5. Spatial patterns
6. Aggregation and growth processes
7. Cellular automata
8. Fractals
9. Miscellaneous topics
10. Concluding session

Literature

- Kaye, Brian H. (1989): A random walk through fractal dimensions. VCH, Weinheim
- Falconer, Kenneth (1990): Fractal Geometry. Mathematical Foundations and Applications. John Wiley&Sons.