Patterns in Nature 8 Fractals

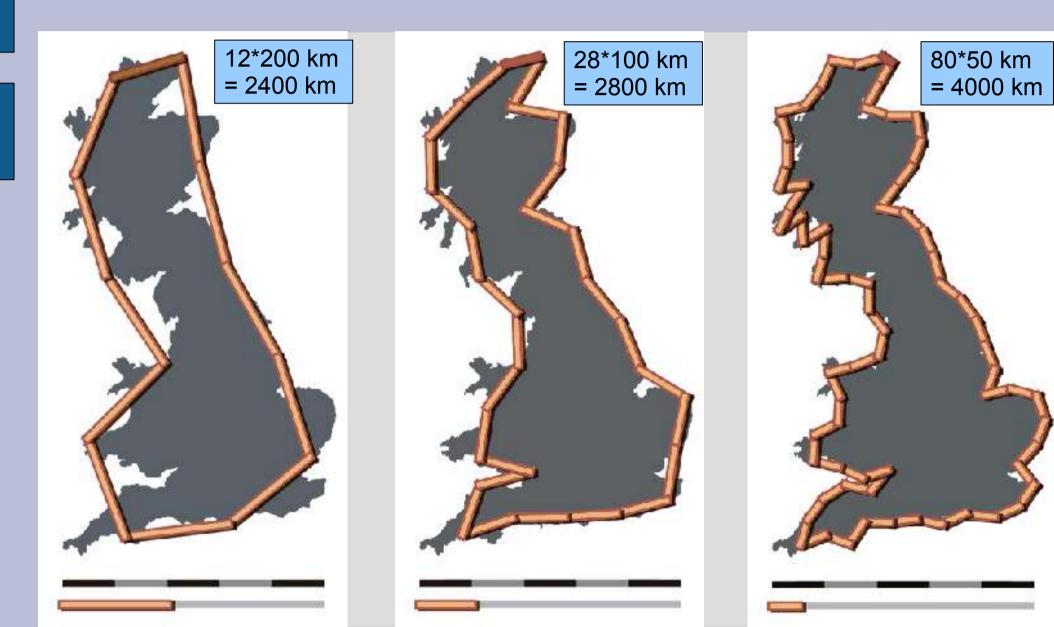
Stephan Matthiesen

How long is the Coast of Britain?

• CIA Factbook (2005): 12,429 km

http://en.wikipedia.org/wiki/Lewis_Fry_Richardson

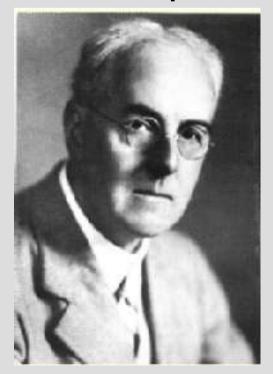
How long is the Coast of Britain?



How long is the Coast of Britain?

 Discussed by Lewis Fry Richardson (1881-1953)
 pioneer of numerical
 Richardson

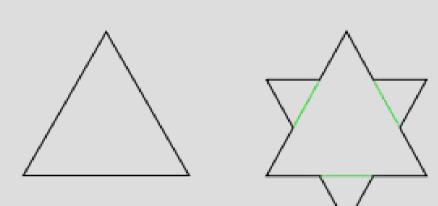
Richardson's "forecast factory"

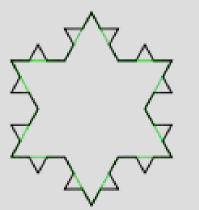


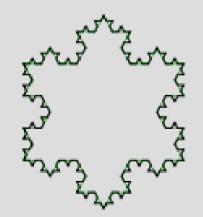


Simple Example: Koch snowflake

- First introduced by Helge von Koch (1904)
- Defined by an iteration rule:
 - Replace the middle line segment by two sides of an equilateral triangle
 repeat infinitely...
- Hausdorff dim.: log(4)/log(3)=1.26







Frequent features of fractals

- F is **self-similar** (at least approximately or stochastically).
- F has a **fine-structure**: it contains detail at arbitrary small scales.
- F has a simple definition.

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- F is obtained through a **recursive** procedure.
- The geometry of F is not easily described in classical terms.
- It is awkward to describe the local geometry of F.
- The size of F is not quantified by the usual measures of length (this leads to the Hausdorff dimension)

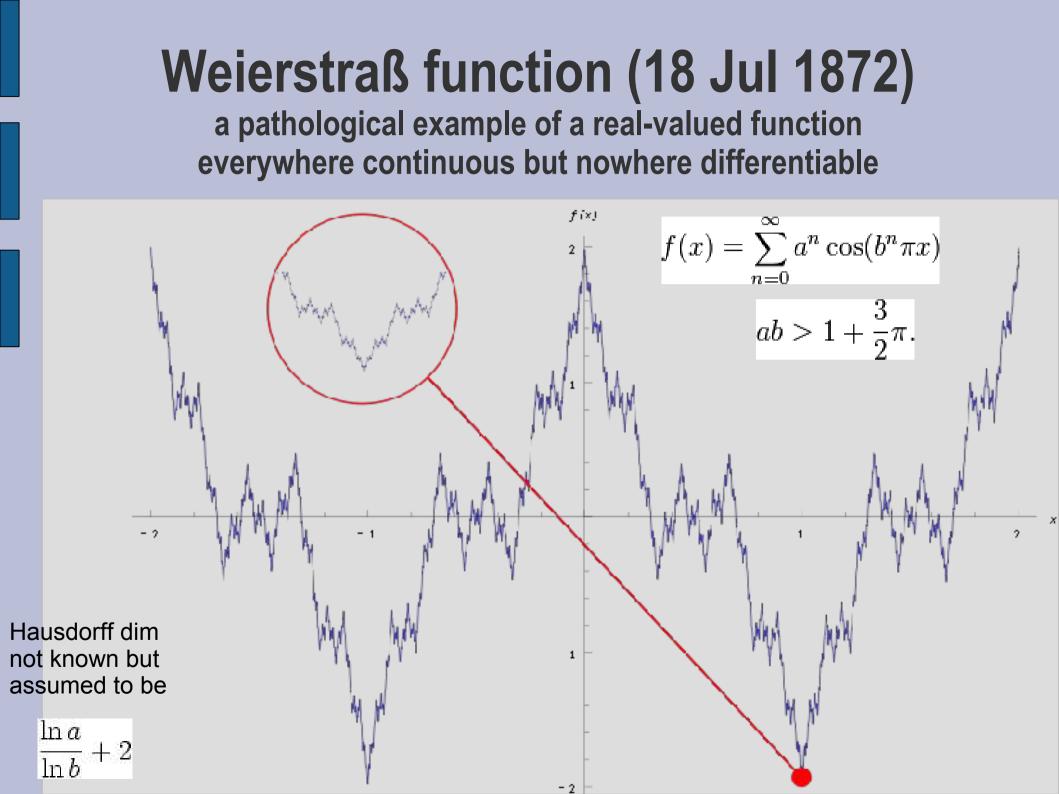
(after Falconer 1990)

Fractals

- Popularized (1967, Nature) by Benoît B. Mandelbrot (1924-2010) he invented the term fractal
- But: Karl Weierstraß (1815-1897) defined continuous nondifferentiable functions





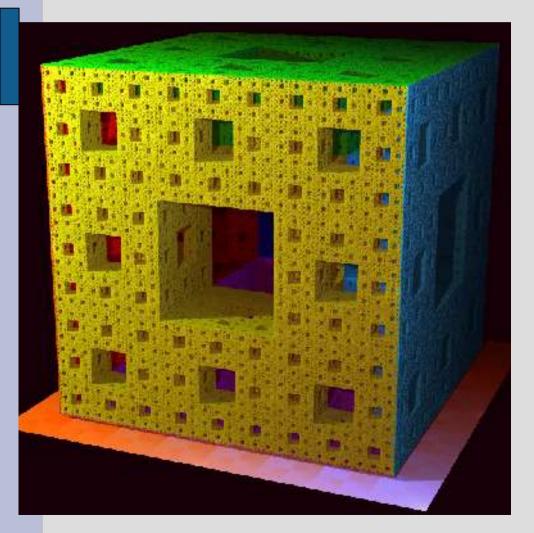


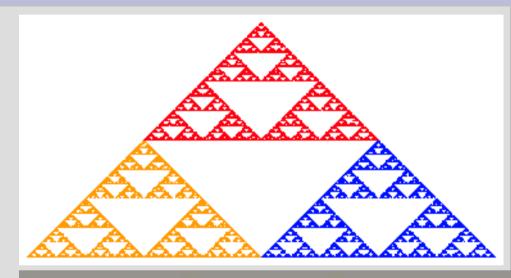
Sierpinski carpet and triangle

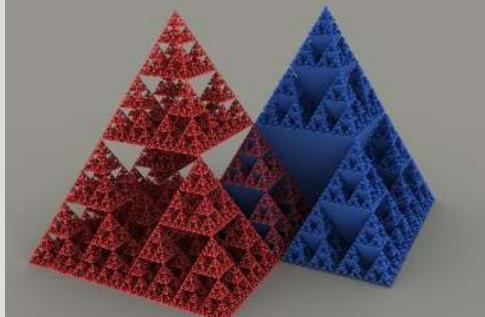
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=log8/log3 =1.8928

Menger sponge and Sierpinski pyramid







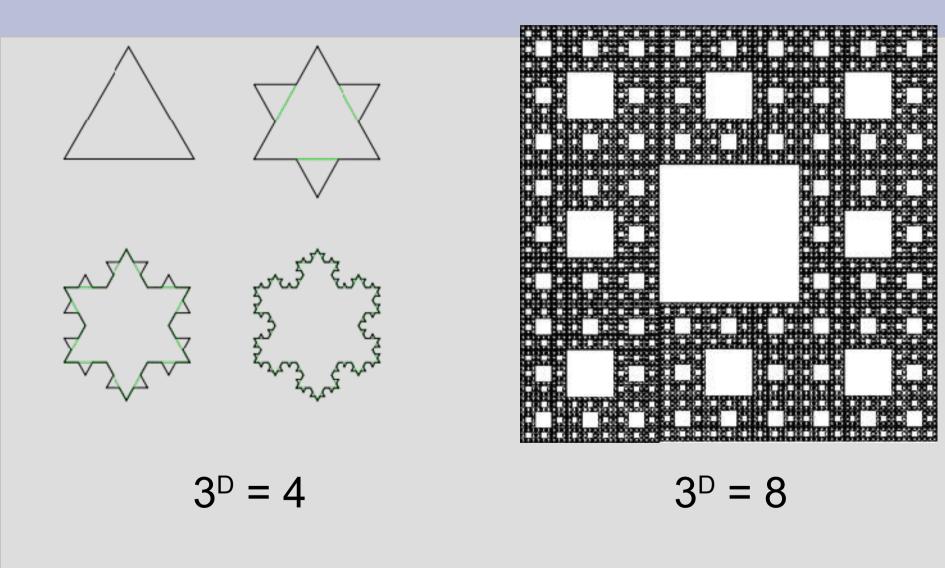
Dimension

- "Normal" dimension (affine dimension of a vector space) defined by number of coordinates that are needed to define a point
- Covering dimension ("topological" dimension) make the cover half as large (a=2)
 - 1-d: make line segments half as long => you need N = 2 = 2¹ times as many (N=2).
 - 2-d: make squares half as large => you need N = 4 = 2² times as many.
 - 3-d: make cubes half as large => you need N = 8 = 2³ times as many.
- Extend this to fractional numbers

a^D=N

D = log(N)/log(a)

Hausdorff dimension



 $D = \log 4 / \log 3 = 1.26$

 $D = \log 8 / \log 3 = 1.8928$

Hausdorff dimension

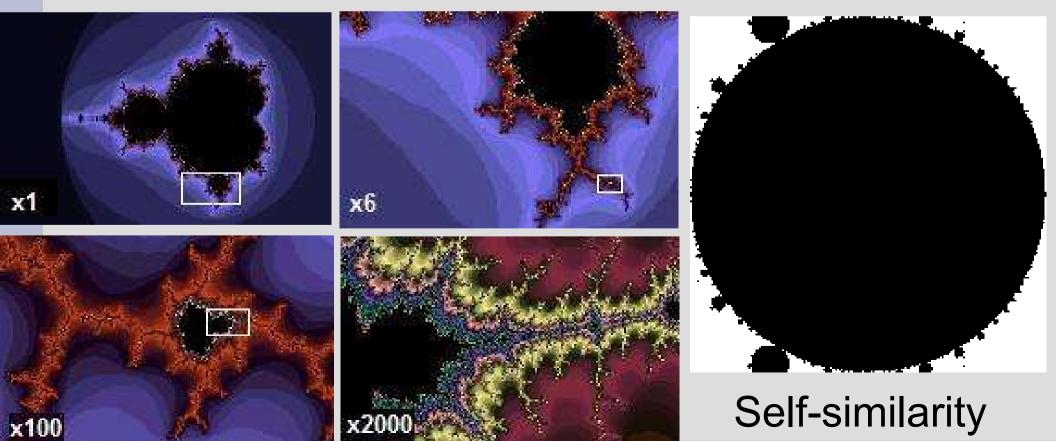
- Defined 1918 by Felix Hausdorff
- Also called
 - Hausdorff-Besicovitch dimension
 - Fractal dimension
 - Capacity dimension
- Can be any real number (unlike the "normal" [integer!] dimension)



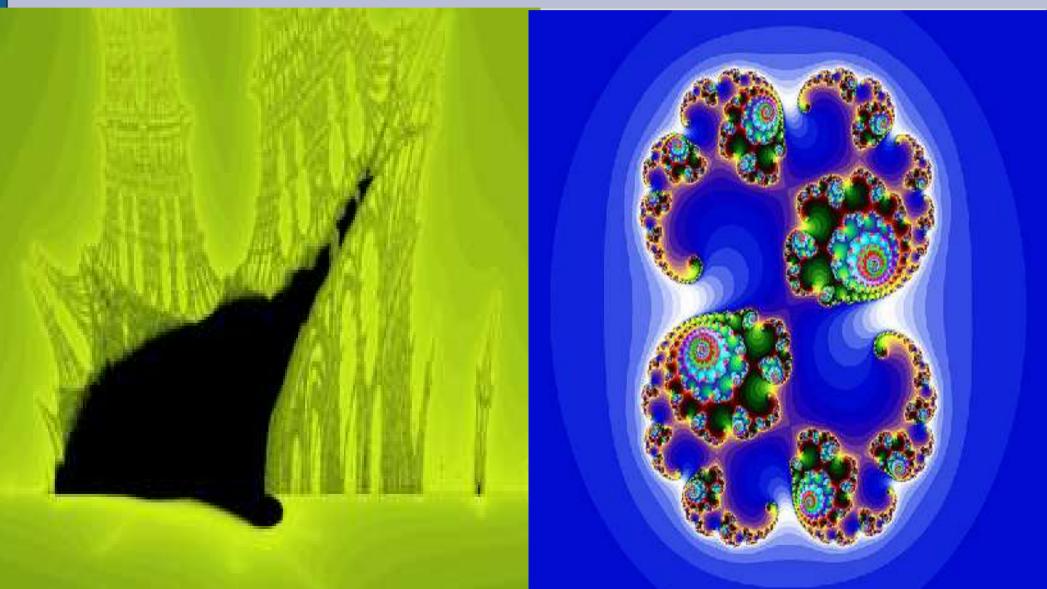
Mandelbrot Set

Mandelbrot Set

- Definition:
 - for every c calculate the iteration $P_c: z \mapsto z^2 + c$
 - if result remains finite, then c belongs to the set



More fractals: "burning ship" and Julia set



Generation of fractals in mathematics

• Escape time fractals:

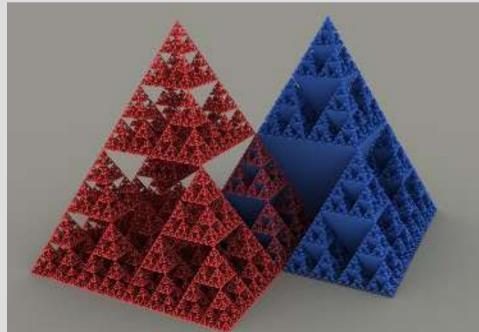
Recurrence relation at each point in space e.g. Mandelbrot set, Julia set

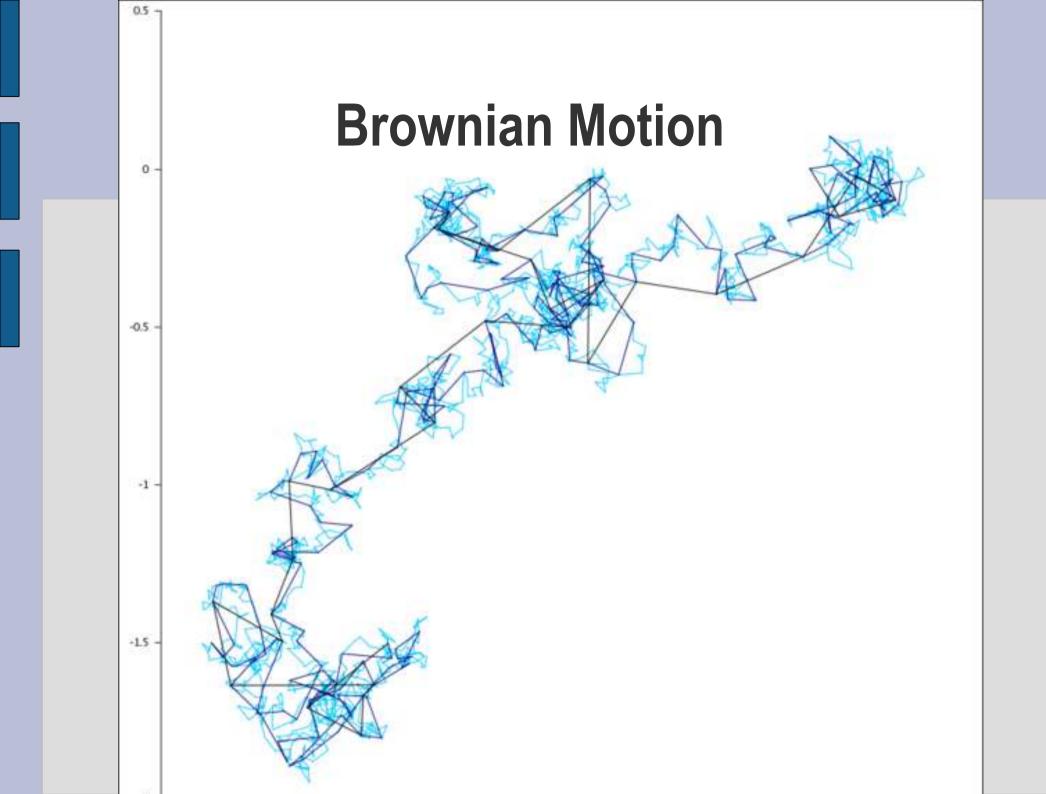
- Iterated function fractals: Fixed geometric replacement rule e.g. Koch snowflake,
- Random fractals:

Stochastic (not deterministic) processes e.g. random walk (Brownian motion), fractal landscapes

Self-similarity

- Exact self-similarity: Fractal appears identical at different scales
- Quasi-self-similarity: Fractal appears approximately identical at different scales
- Statistical self-similarity: Some numerical or statistical measure is preserved across scales





Snowflakes

Wilson Bentley (1865-1931)



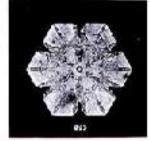






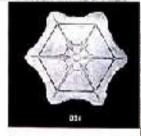


















Manganese oxide dendrites





Bransley's fern (computer generated)

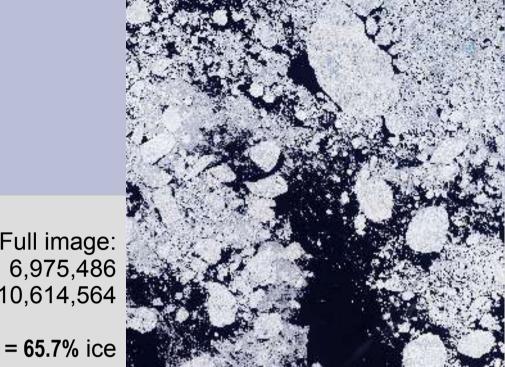


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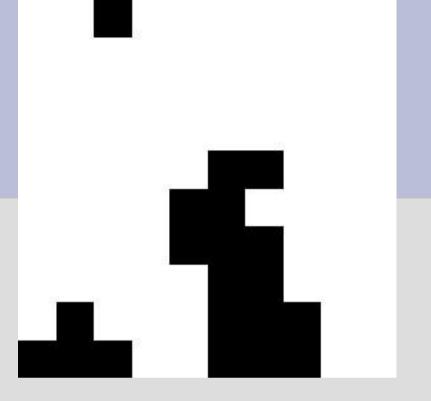
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Full image: 6,975,486 / 10,614,564



80 ice pixels out of 100 = 80% ice cover

Dimension (using the 10x10 and 100x100 images):

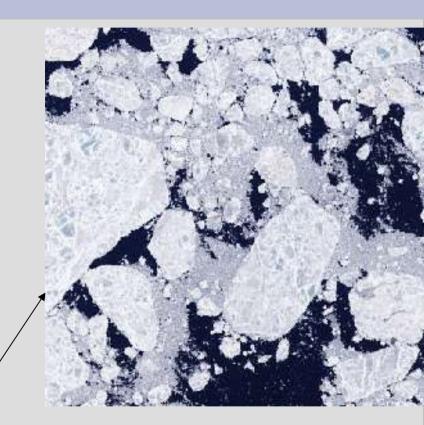
a = 10 N = 7010/80 = 87.625

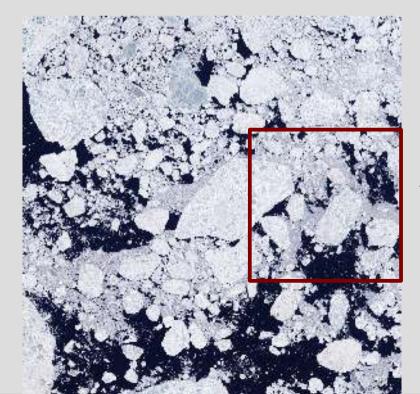
dim = log (N) / log (a) = 1.94

7010 ice pixels / 10000 in total = 70.1% ice cover









Frequent features of fractals

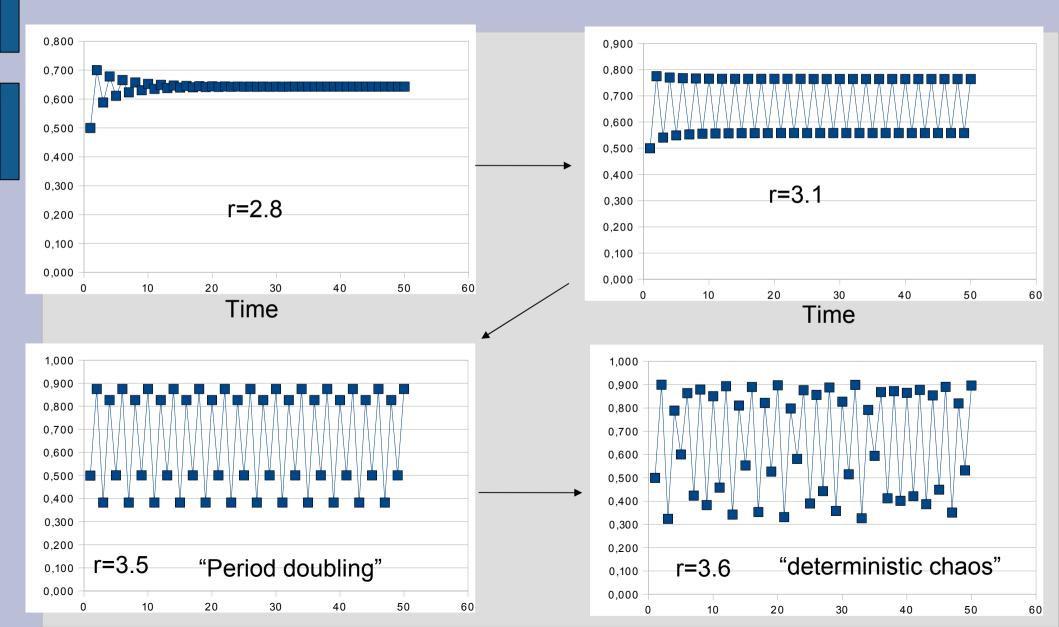
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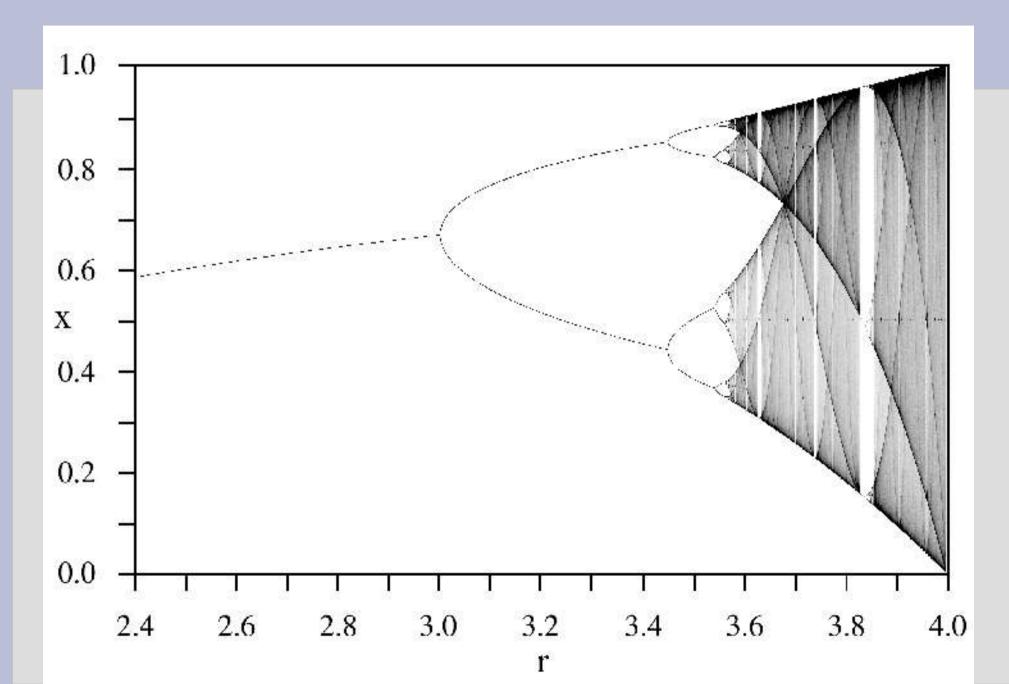
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(after Falconer 1990)

The link to Chaos I: The logistic map

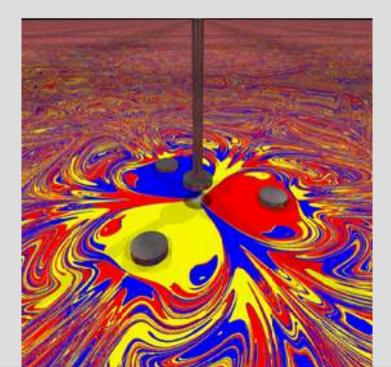


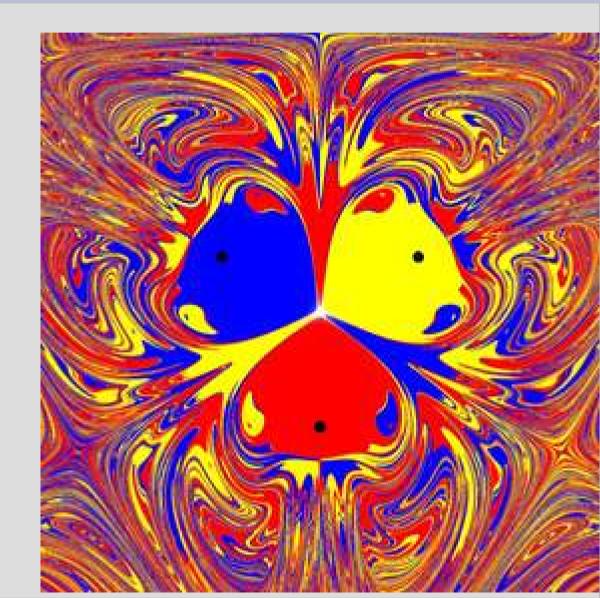
Bifurcation diagram



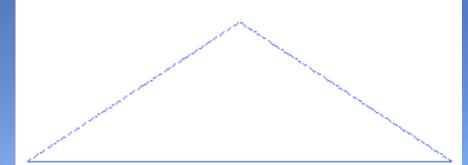
The link to Chaos II: The magnetic pendulum

For each starting point, calculate the final resting position (over one of the three magnets) Then colour the starting point accordingly **Result: a fractal!**





Fractal landscape



Random midpoint displacement

General remarks

- Fractal geometry describes shapes
- It does not explain mechanisms how these shapes grow (although it can provide some constraints for possible mechanisms)
- It not inform us about the (evolutionary) function of an (biological) objects

Patterns in Nature Outline

- 1. Introduction
- 2. Waves and oscillations
- 3. Regularity and chaos
- 4. Animal cooperation
- 5. Spatial patterns
- 6. Aggregation and growth processes
- 7. Cellular automata
- 8. Fractals
- 9. Miscellaneous topics
 10. Concluding session





Literature

- Kaye, Brian H. (1989): A random walk through fractal dimensions. VCH, Weinheim
- Falconer, Kenneth (1990): Fractal Geometry. Mathematical Foundations and Applications. John Wiley&Sons.