# Atmospheric Dynamics Revision topics

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# Main structure of course

- Expressing Newton's laws for a fluid on a rotating sphere
- Consequence of scale (Rossby Number) gives geostrophic and higher approximations for wind in terms of pressure (or geopotential)
- Variation of wind with height
  - In free atmosphere – "thermal" wind
  - In boundary layer, eddy friction, and Ekmann spiral
- Circulations in the tropics (introduced via "dishpan" results)
- Notions of vorticity, divergence, and potential vorticity
- Use of those concepts in

# Newton's Laws on sphere

Preliminaries: hydrostatic equation, potential temperature and static stability

Material Derivative

$$s = s(t, x, y, z) \qquad \delta s = \frac{\partial s}{\partial t}\delta t + \frac{\partial s}{\partial x}\delta x + \frac{\partial s}{\partial y}\delta y + \frac{\partial s}{\partial z}\delta z$$

$$\frac{Ds}{Dt} = \left(\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} + w\frac{\partial s}{\partial z}\right)$$

Rate of change of momentum of lump = surface integral of Pressure forces + vol integral of gravity force.

Leads (after applying Gauss's theorem) to

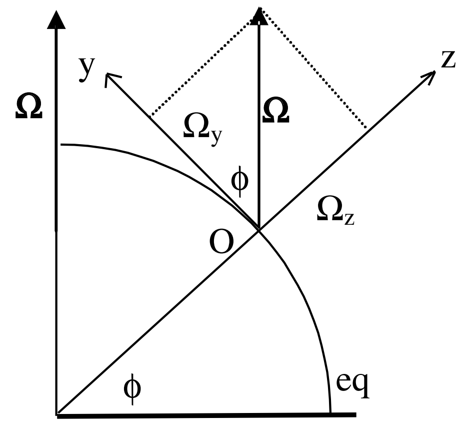
$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g}_t$$

# Effect of rotating frame

$$\left(\frac{D}{Dt}\right)_f = \left(\frac{D}{Dt}\right)_r + \boldsymbol{\Omega} \times$$

$$\left(\frac{D}{Dt}\right)\boldsymbol{v} + 2\,\boldsymbol{\Omega} \times \boldsymbol{v} = -\frac{1}{\rho}\nabla\, p + g$$

$$\wedge\, \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Omega_x & \Omega_y & \Omega_z \\ u & v & w \end{vmatrix}$$

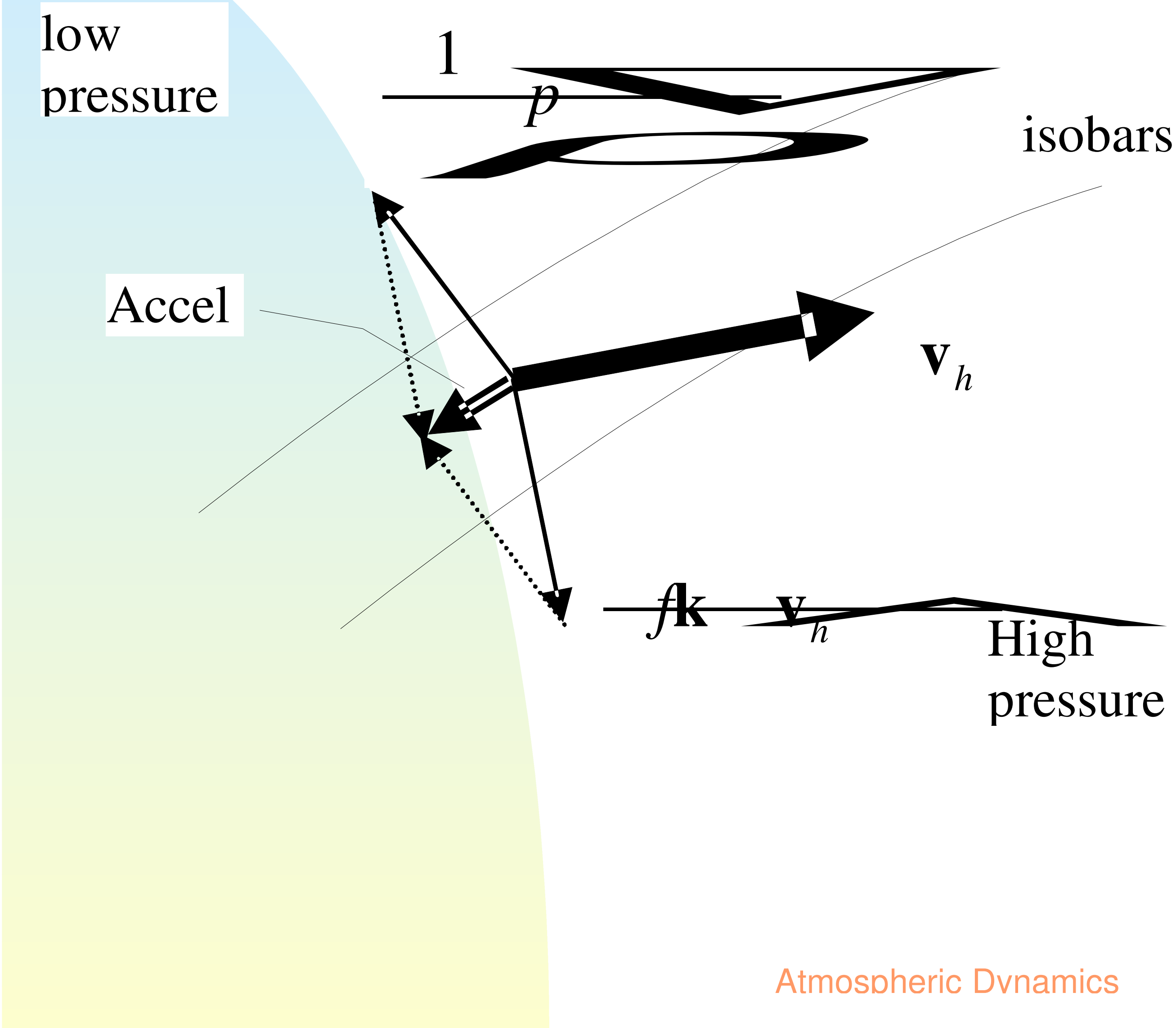



$$\frac{Du}{Dt} + 2\Omega w \cos\phi - 2\Omega v \sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} + 0 + 2\Omega u \sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{Dw}{Dt} + 0 - 2\Omega u \cos\phi = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

# Summary of forces



# Rossby No. and geostrophy

Rossby number, $R_o$, is the ratio of typical values of the relative acceleration and Coriolis acceleration

$R_o = (U^2/L)/(fU) = U/(Lf)$

U=10ms$^{-1}$, L=1000km, f~10$^{-4}$ gives $R_o$=1/10

Thus relative accel << Coriolis accel

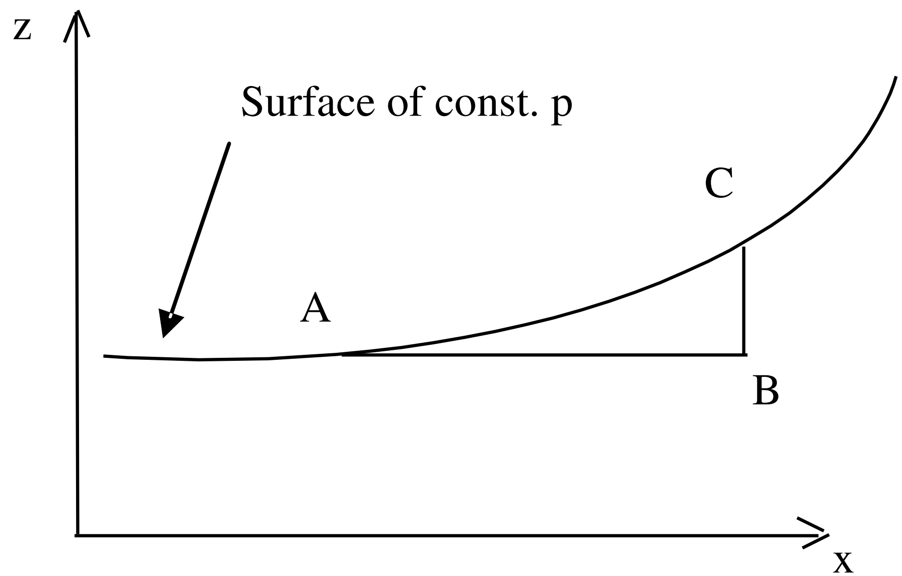
Define the *geostrophic wind*, $\mathbf{v}_g$ for which s.c.f. <u>exactly</u> balances s.p.g.f.

$$-f\mathbf{k}\wedge v_g - \frac{1}{\rho}\nabla p \equiv 0 \qquad\qquad \mathbf{v}_g \equiv \frac{1}{\rho f}\mathbf{k}\wedge\nabla p$$

Higher approx to true wind can be found from the pressure field if acceleration can be estimated

- directly (as in gradient wind – steady circular flow)
- by using geostrophic approximation in the terms for acceleration

# Pressure as vertical co-ordinate




$$\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_{t,y,z} \equiv \frac{1}{\rho}\underset{\delta x\to 0}{Lim}\frac{\delta p}{\delta x} = -\frac{1}{\rho}\underset{\delta x\to 0}{Lim}\frac{\rho\delta\varphi}{\delta x} = -\left(\frac{\partial \varphi}{\partial x}\right)_{t,y,p}$$

$$\frac{D}{Dt}\mathbf{v}_h + f\mathbf{k}\wedge\mathbf{v}_h = -\nabla_p\varphi$$

# Thermal wind

Difference of winds at two pressure levels is *Thermal Wind* $\mathbf{v}_T$

$$\mathbf{v}_T = \mathbf{v}_{g2} - \mathbf{v}_{g1} = \frac{g}{f}\mathbf{k} \wedge \left(\nabla_p z_2 - \nabla_p z_1\right)$$

$$= \frac{g}{f}\mathbf{k} \wedge \nabla_p z'$$

Integrated hydrostatic equation gives

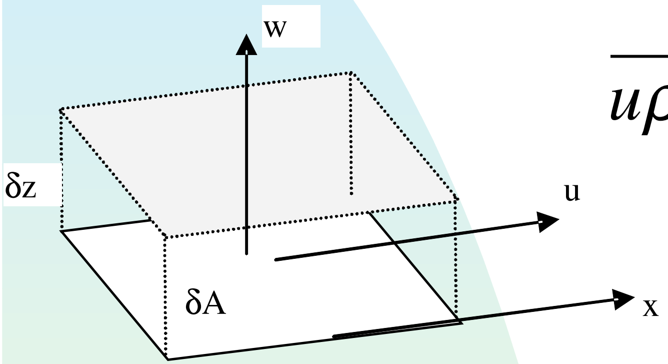
$$z' = \frac{R\overline{T}}{g}\ln\frac{p_1}{p_2}$$

$$\mathbf{v}_T = \frac{R}{f}\left(\ln\frac{p_1}{p_2}\right)\mathbf{k} \wedge \nabla\overline{T}$$

# Eddy friction

Mean $$\bar{s} = \frac{1}{2Q}\int_{t_o - Q}^{t_o + Q} s\,dt$$

eddy $$s' \equiv s - \bar{s}$$




$$\overline{u\rho w} = \rho\overline{(\bar{u}+u')(\bar{w}+w')} = \rho\overline{(\bar{u}\bar{w}+u'\bar{w}+\bar{u}w'+u'w')}$$

$$= \rho(\bar{u}.\bar{w}) + \rho\left(\overline{u'w'}\right)$$

Upward flux of x-mom^t due to the eddies = $$\tau_{xz} = \rho\left(\overline{u'w'}\right)$$

Accel due to the eddies = $$-\frac{1}{\rho}\frac{\partial \tau}{\partial z}$$

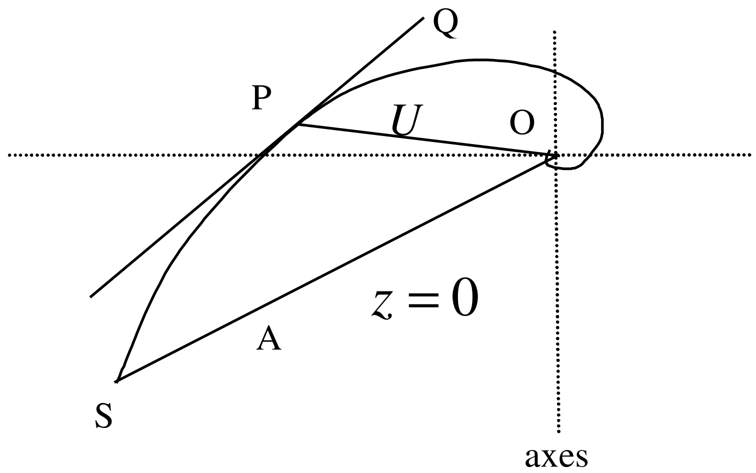
Introduce eddy diffusion coeff. $$\tau_{xz} = -\rho K \frac{\partial \bar{u}}{\partial z}$$

# Ekmann Spiral

- Steady unaccelerated flow
- Uniform geostrophic wind in vertical and horizonal
- Variations of density small in bottom km

$$f\mathbf{k}\wedge\overline{\mathbf{v}}_h = -\frac{1}{\rho}\nabla_h p + K\left(\frac{\partial^2\overline{\mathbf{v}}_h}{\partial z^2}\right) \qquad f\mathbf{k}\wedge\mathbf{v}_a = +K\left(\frac{\partial^2\mathbf{v}_a}{\partial z^2}\right)$$

Shape of ageostrophic component



# Vorticity Equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial y}$$

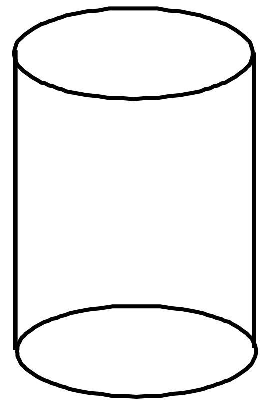
$$\left(\frac{D}{Dt}\right)_h(\zeta_{rel}+f)=-(\zeta_{rel}+f)div_h\mathbf{v}_h$$

$$\left(\frac{D}{Dt}\right)_h(\zeta_{abs})=-(\zeta_{abs})div_h\mathbf{v}$$

$$\zeta_{rel}=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$$\zeta_{abs}=\zeta_{rel}+f$$

# Potential vorticity




$$\left(\frac{D}{Dt}\right)_p (\zeta_{abs}) = \zeta_{abs}\frac{\partial \varpi}{\partial p}$$

$$\delta p = p_1 - p_2$$

$$\frac{1}{\delta p}\frac{D\delta p}{Dt} = \frac{\partial \varpi}{\partial p}$$

$$\left(\frac{D}{Dt}\right)_p (\zeta_{abs}) - \frac{\zeta_{abs}}{\delta p}\frac{D\delta p}{Dt} = 0$$

$$\left(\frac{D}{Dt}\right)_p \frac{\zeta_{abs}}{\delta p} = 0$$

$$\frac{\zeta_{abs}}{\delta p}$$

**a.k.a. the potential vorticity is conserved**

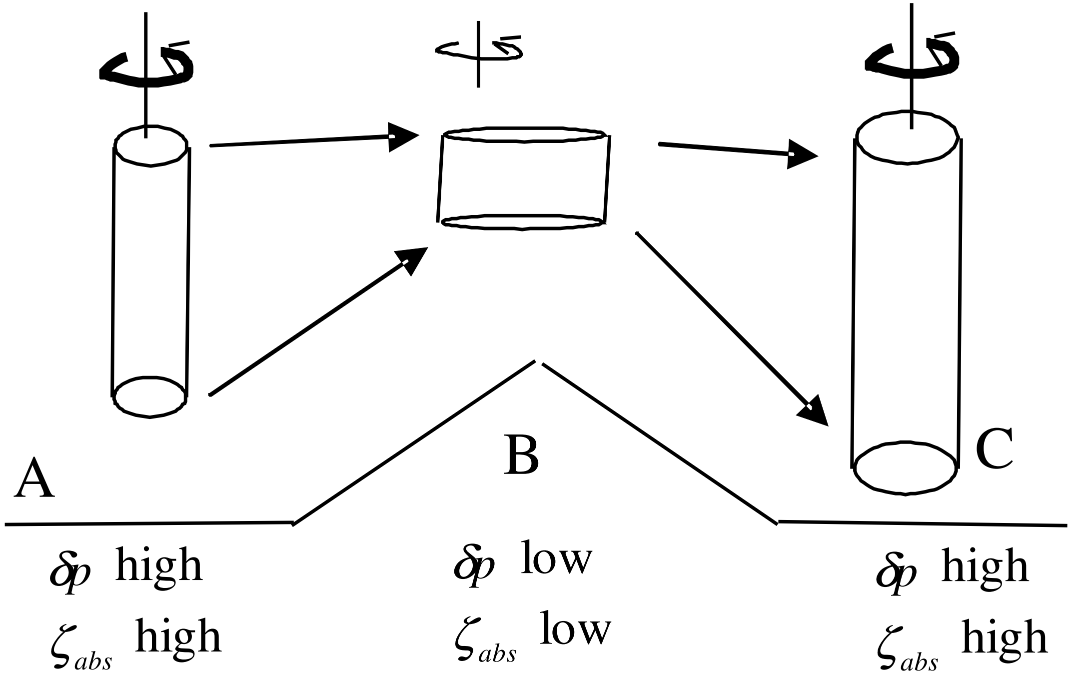
**In frictionless flow**

# Flow over mountain

At A v=0 $\frac{\partial u}{\partial y} = 0$

$\zeta_{rel} = 0$

$\zeta_{abs} = f$




At B v=0 δp small $\zeta_{abs} < f$ $\zeta_{rel} < 0$ $\frac{\partial v}{\partial x} < 0$

# f-plane and β-plane approximations

- Both use tangent plane approx.
- For f-plane, f=constant=$f_o$
- For β-plane, f is made a linear function of y

$$f = f_0 + \beta(y - y_0) \qquad \beta = \left(\frac{\partial f}{a\partial\phi}\right)_0 = \frac{2\Omega\cos\phi_0}{a}$$

**for** $\phi_0 = 45^o N$ $$\beta = \frac{10.3\times10^{-5}s^{-1}}{6371km} = 1.62\times10^{-11}m^{-1}s^{-1}$$

# Stream function

$$u_g = -\frac{\partial \psi}{\partial y} \qquad v_g = +\frac{\partial \psi}{\partial x} \longrightarrow \mathbf{v}_g = \mathbf{k} \wedge \nabla_p \psi$$

$$\zeta_{rel} \sim \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla_p^2 \psi$$

**ψ is the *streamfunction***

# The Quasi-geostrophic equations

The complete evolution of the flow is governed by the two equations

$$\left(\frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}\right)_p \nabla_p{}^2\psi+v_g\beta=f_0\frac{\partial\varpi}{\partial p}$$

$$\left(\frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}\right)_p \frac{\partial\psi}{\partial p}+\frac{\Gamma_0 R}{f_0 p}\varpi=0$$