Atmospheric Dynamics Lect.16: Baroclinic Instability

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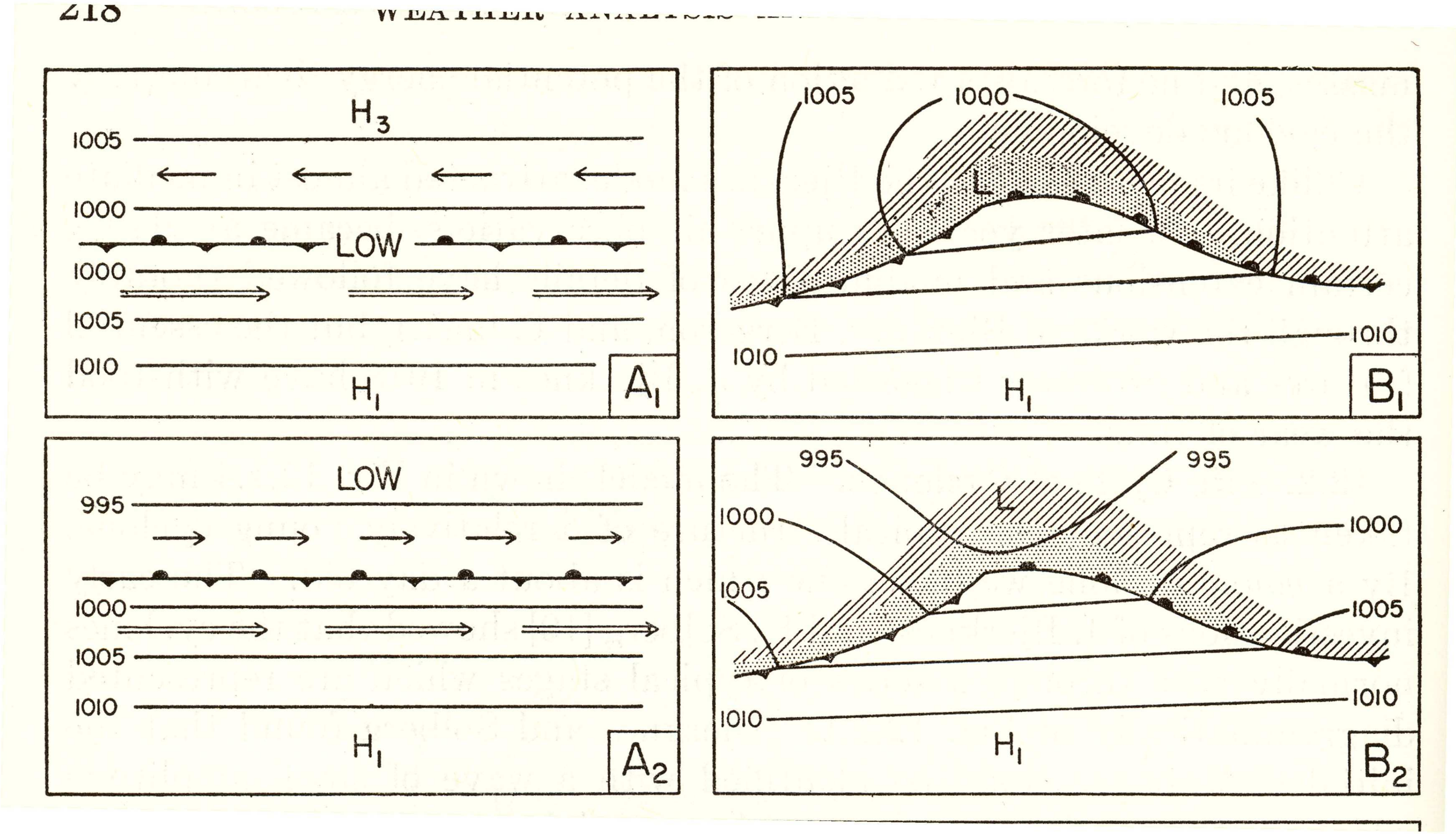
Scope of Lecture

- Observed growing waves
- Adaptation of equations to f-plane and incompressible fluid with basic shear
- Conditions for growth: short wave cut-off
- Selection of fastest growing wave
- Structure of the growing wave
- Implied limits to forecasting

Cyclone – early stages



Cyclone – later stages

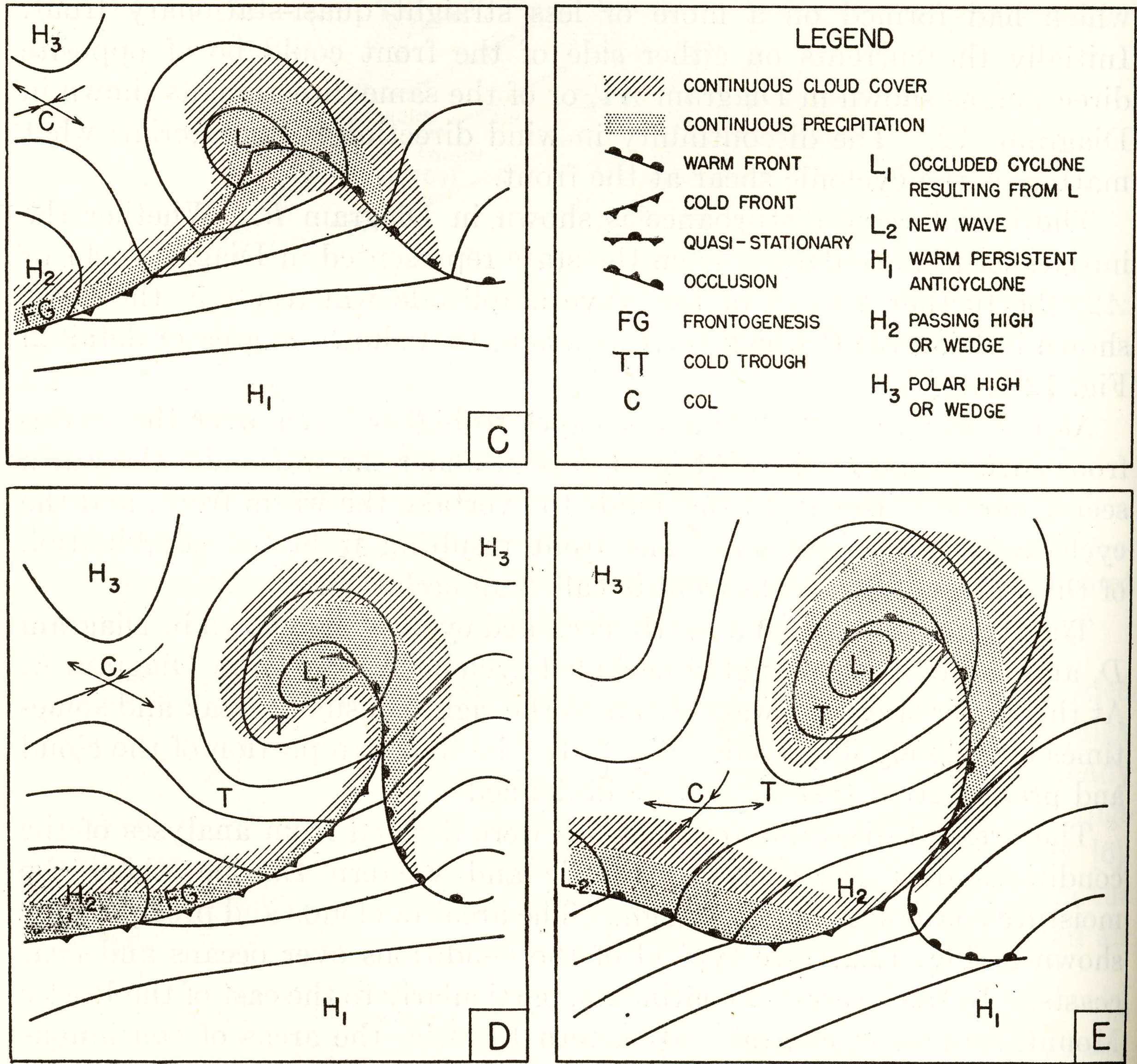


FIG. 12.2.1. The life cycle of extratropical cyclones. (*Mainly after J. Bjerknes and*

Incompressible flow on f-plane

For compressible flow on tangent plane we have

$$\left(\frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}\right)_p q=0 \qquad q=\nabla^2\Psi+f+\frac{f_0^2}{N^2}\left(\frac{\partial^2\Psi}{\partial z^{*2}}-\frac{1}{H}\frac{\partial\Psi}{\partial z^*}\right)$$

For f-plane $\quad f=f_0$

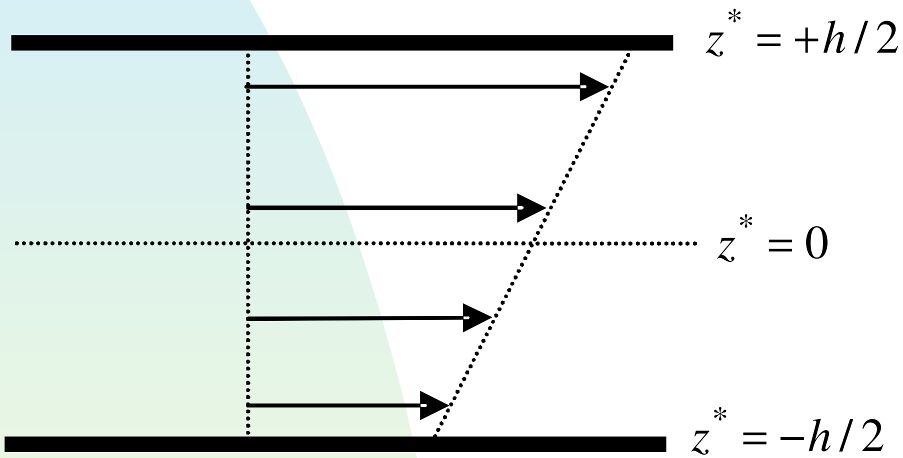
For incompressible flow $\quad H\rightarrow\infty$

giving

$$q=\nabla^2\Psi+f_o+\frac{f_0^2}{N^2}\left(\frac{\partial^2\Psi}{\partial z^{*2}}\right)$$

Flow configuration

As before denote basic flow by u_0



Simple shear between lids at z^*=+/- h/2

$$u_0 = u_{00} + \Lambda z^*$$

There is an associated N-S temperature gradient

The upper lid is a crude representation of stratosphere

Linearised equation and wavelike solutions

set $(u, v) = (u_0, 0) + (u', v')$ and $\psi = \psi_0 + \psi'$

The q-g potential vorticity equation linearises to

$$\left\{ \frac{\partial}{\partial t} + \left(u_{00} + \Lambda z^* \right) \frac{\partial}{\partial x} \right\} \left\{ \nabla^2 \psi' + \frac{f_0^2}{N^2} \left(\frac{\partial}{\partial z^*} \right)^2 \psi' \right\} = 0$$

giving $$\nabla^2 \psi' + \frac{f_0^2}{N^2} \left(\frac{\partial}{\partial z^*} \right)^2 \psi' = 0$$

Seek solutions like $$\psi' \propto \exp i \left(\sigma t + \lambda x + \mu y + \nu z^* \right)$$

Requires $$-\left(\lambda^2 + \mu^2 \right) - \frac{f_0^2}{N^2} \nu^2 = 0$$

Vertical structure and boundary conditions

We see that $\nu = \pm i\alpha$

with α wholly real and positive and given by

$$\alpha \equiv \frac{N}{f_0}\sqrt{\left(\lambda^2 + \mu^2\right)}$$

α is a scaled horizontal wavelength

To find A and B we impose boundary conditions, but these involve w, so we need the thermodynamic equation

Thermodynamic equation for incompressible flow

For compressible flow we had
$$\frac{\partial \psi}{\partial z^*} = \frac{R}{f_0 H_{00}} T$$

with $H_{00} = RT_{00}/g$ so that
$$\frac{\partial \psi}{\partial z^*} = \frac{g}{f_0 T_{00}} T$$

For incompressible flow the same equation holds:-

$$\frac{\partial \psi}{\partial z^*} = \frac{g}{f_0 T_{00}} T$$

The thermodynamic equation is virtually unchanged:-

$$\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right)_p \frac{\partial \psi}{\partial z^*} + w^* N^2 = 0$$

Linearised form

Linearised thermodynamic equation is

$$\left(\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}\right)_p\frac{\partial \psi'}{\partial z^*}+\left(v'\frac{\partial}{\partial y}\right)_p\frac{\partial \psi_0}{\partial z^*}+w^{*'}N^2=0$$

$$\left(v'\frac{\partial}{\partial y}\right)_p\frac{\partial \psi_0}{\partial z^*}=v'\frac{\partial u_0}{\partial z^*}=\Lambda v'=\Lambda\frac{\partial \psi'}{\partial x}$$

$$\left(\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}\right)_p\frac{\partial \psi'}{\partial z^*}+\Lambda\frac{\partial \psi'}{\partial x}+w^{*'}N^2=0$$

Application to boundary conditions

$w^* = 0$ at $z^* = \pm h/2$

At z^*=+h/2

$$\left(\frac{\partial}{\partial t} + \frac{h\Lambda}{2}\frac{\partial}{\partial x}\right)_p \frac{\partial \psi'}{\partial z^*} + \Lambda \frac{\partial \psi'}{\partial x} = 0$$

At z^*=-h/2

$$\left(\frac{\partial}{\partial t} - \frac{h\Lambda}{2}\frac{\partial}{\partial x}\right)_p \frac{\partial \psi'}{\partial z^*} + \Lambda \frac{\partial \psi'}{\partial x} = 0$$

On substitution for ψ'

$$\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

in which the r_{ij} involve α, λ, μ, σ

Equation for time variation

Last equation being homogeneous, consistency requires

$$\begin{vmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{vmatrix} = 0$$

This expands to

$$\left(\frac{\sigma}{\lambda} + u_{00}\right)^2 = \Lambda^2 h^2 \left[\frac{1}{4} - \frac{\coth(\alpha h)}{\alpha h} + \frac{1}{(\alpha h)^2}\right]$$

So the horizontal scale fixes the time-dependence

Note rhs apparently can be positive or negative

Neutral case

We can show R.H.S. positive for $\alpha > \alpha_{crit}$ (short-wavelengths)

Then σ is wholly real so write

$$\sigma = \sigma_r + i\sigma_i = -c\lambda + i0$$

$$\left(u_{00} - c\right) = \pm\sqrt{R.H.S}$$

Turns out that wave amplitudes are constant in time

and disturbance is confined near to a lid.

Wavespeed characteristic of velocity near the relevant lid

Growing case

Reminder

$$\left(\frac{\sigma}{\lambda}+u_{00}\right)^2=\Lambda^2h^2\left[\frac{1}{4}-\frac{\coth(\alpha h)}{\alpha h}+\frac{1}{(\alpha h)^2}\right]$$

Small α (waves longer than critical) RHS is negative

$$\left(\frac{\sigma}{\lambda}+u_{00}\right)$$

is wholly imaginary

Set

$$\sigma=\sigma_r+i\sigma_i=-c\lambda+i\sigma_i$$

$$\left(u_{00}-c\right)=0$$

$$\left(\frac{\sigma_i}{\lambda}\right)=\pm\Lambda h\sqrt{\left|\frac{1}{4}-\frac{\coth(\alpha h)}{\alpha h}+\frac{1}{(\alpha h)^2}\right|}$$