Chapter 15: Baroclinic Rossby waves

The theory developed in the previous chapter was clearly highly restrictive, in that we have no real reason for expecting the fluid to behave barotropically nor – what amounts to the same thing – for expecting the vertical velocities to be zero. Accordingly in this chapter we relax some of these restrictions.

In this chapter we will consider long waves in the westerlies again, but unlike in the previous chapter we shall allow vertical velocities, which means that there can be horizontal temperature gradients and variations in the flow with height. Hence the flow is not barotropic. Flows which are not barotropic are known as *baroclinic*.

It turns out that we can explain several features of the variations throughout the troposphere and stratosphere of the characteristics of the long waves. As the pressure in the stratosphere changes by a couple of orders of magnitude, there are some disadvantages in using pressure as vertical co-ordinate, as this crams the stratosphere into a tiny portion of the vertical axis. Accordingly we need a more suitable co-ordinate system.

Log pressure co-ordinates

In this co-ordinate system we shall use a multple of $-\ln p$ as the vertical co-ordinate. As pressure decreases approximately exponentially with height, this variable is approximately proportional to height. Indeed in an isothermal atmosphere it would be exactly proportional to height. This makes it useful to define the new vertical co-ordinate as $z^* = H_{00}\ln\left(\frac{p_{00}}{p}\right)$, where $H_{00} = \frac{RT_{00}}{g}$ and p_{00} and T_{00} are reference pressures and temperatures, which can be arbitrarily chosen, but which are normally taken to be $1000hPa$ and 250K respectively. This means that $z^* = 0$ at $p = 1000hPa$, and $H_{00} = 7.3\text{km}$. The symbol z^* has been chosen to remind us that it is like height (z) but not quite the same (hence the asterisk). At all stages it should be kept in mind that it is really just a way of re-labelling the pressure.

We define the vertical velocity, w^*, in the obvious way by $w^* = \frac{Dz^*}{Dt}$, so that

$$w^* = \frac{Dz^*}{Dt} = -H_{00}\frac{1}{p}\frac{Dp}{Dt} = -\frac{H_{00}\omega}{p}$$

Eq 1

Vorticity equation in the ln(p) co-ordinates

We need to transform the vorticity equation from eq 4 of chapter 13. The left hand side looks the same as before, and all we have to do is re-work $f_0 \frac{\partial \omega}{\partial p}$. From Eq 1 we have $\omega = -\frac{p}{H_{00}} w^* = -\frac{p_{00}}{H_{00}} w^* \exp\left(-z^* / H_{00}\right)$.

Also $\frac{\partial}{\partial p} = \frac{\partial \ln p}{\partial p} \frac{\partial}{\partial \ln p} = -\frac{H_{00}}{p} \frac{\partial}{\partial z^*}$

Eq 2

Together these lead to $f_0 \frac{\partial \omega}{\partial p} = f_0 \left(\exp \frac{z^*}{H_{00}} \right) \frac{\partial}{\partial z^*} \left\{ w^* \exp \frac{-z^*}{H_{00}} \right\}$.

Inserting this into the vorticity equation gives

$$\left(\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y} \right) \nabla^2 \psi + \beta \frac{\partial \psi}{\partial x} = f_0 \left(\exp \frac{z^*}{H_{00}} \right) \frac{\partial}{\partial z^*} \left\{ w^* \exp \frac{-z^*}{H_{00}} \right\}$$

Eq 3

with $\psi = \frac{\varphi}{f_0}$ and $u_g = -\frac{\partial \psi}{\partial y}$ and $v_g = \frac{\partial \psi}{\partial x}$.

Eq 4

The thermodynamic equation in ln(p) co-ordinates

As was discussed in chapter 13, we shall also need the thermodynamic equation to completely specify the system. As the first step in transforming this to the log pressure co-ordinates, we note that

$$\frac{\partial \psi}{\partial z^*} = -\frac{p}{f_0 H_{00}} \frac{\partial \varphi}{\partial p} = \frac{1}{f_0 H_{00}} \frac{p}{\rho} = \frac{R}{f_0 H_{00}} T ,$$

Eq 5

where we have made use of Eq 2, Eq 4, the hydrostatic equation and the gas law. The right hand side is simply a multiple of the temperature. (You should note how Eq 5 is related to the thermal wind equation.)

In p co-ordinates, one form of the thermodynamic equation was shown to be (see chapter 13 eq 6)

$$\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right)_p T + \varpi \frac{T}{\theta} \frac{\partial \theta}{\partial p} = 0$$

Eq 6

It is easy to show that the final term can be re-written

$$w^*\frac{T}{\theta}\frac{\partial\theta}{\partial z^*}=w^*\frac{T}{g}N^2=w^*\frac{H}{R}N^2\,,$$

where $N\equiv\sqrt{\dfrac{g}{\theta}\dfrac{\partial\theta}{\partial z^*}}$ is a Brunt-Vaisaila frequency (see chapter 2) in these co-ordinates and H is the local pressure scale-height

Finally, using Eq 5 allows Eq 6 to be written

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)_p\frac{\partial\psi}{\partial z^*}+w^*\frac{H}{H_{00}}N^2=0$$

or

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)_p\frac{\partial\psi}{\partial z^*}+w^*N^{*2}=0$$

Eq 7

where we have written $N^*=N\sqrt{\dfrac{H}{H_{00}}}$.

Quasi-geostrophic potential vorticity

Substituting for w^* from Eq 7 into Eq 3 leads to

$$\boxed{\left(\frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}\right)_p q_g=0}$$

Eq 8

with

$$q_g=\nabla^2\psi+f+\left(\exp\frac{z^*}{H_{00}}\right)\frac{\partial}{\partial z^*}\left\{\frac{f_0^2}{N^{*2}}\left(\exp\frac{-z^*}{H_{00}}\right)\frac{\partial\psi}{\partial z^*}\right\}$$

Eq 8 tells us that a quantity q_g is conserved following the geostrophic motion on a constant pressure surface. This is clearly the quasi-geostrophic potential vorticity. It is actually straightforward to deduce this form directly from the form given in pressure co-ordinates in chapter 13.

Application to planetary-scale waves.

We are now in a position to rework the problem of Rossby waves in a basic flow, but this time without the restriction to barotropic motion. We will still need to linearise the problem by considering small perturbations to a simple basic flow. In principle the basic flow could be more complicated this time, for instance it could vary with height. However, to keep this course to manageable we will confine ourselves to a basic flow which is uniform in time and space, so u_0 will be a constant. As in chapter 13 we will use a subscript 0 to denote the basic flow and the dash to denote the small perturbations to it.

To keep the analysis relatively simple we shall assume that for our flow the modified Brunt-Vaisaila frequency N^* is constant in space and time.

It is readily seen that Eq 8 linearises to

$$\left(\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}\right)\left[\nabla^2\psi'+\frac{f_0^2}{N^{*2}}\frac{\partial^2\psi'}{\partial z^{*2}}-\frac{f_0^2}{H_{00}N^{*2}}\frac{\partial\psi'}{\partial z^*}\right]+\beta\frac{\partial\psi'}{\partial x}=0.$$

Applying the standard method of solving this linear, separable partial differential equation, we seek solutions of the form

$$\psi'=A\exp i(\sigma t+\lambda x+\mu y+nz).$$

In this n is a (possibly complex) number (not an integer). Substituting gives

$$i(\sigma+u_0\lambda)\left[-(\lambda^2+\mu^2)-\frac{f_0^2}{N^{*2}}n^2-i\frac{f_0^2}{H_{00}N^{*2}}n\right]+i\beta\lambda=0$$

Now we expect to get wavelike solutions in the horizontal, so λ and μ are real. This means that $n^2+i\dfrac{n}{H_{00}}$ must be real also. Write $n=\nu+in_i$, where we are expecting both ν and n_i to be real. (We have used ν rather than say n_r as this will then correspond to a wavelike solution in the vertical and will be consistent with our use of Greek letters for the wavenumbers of the horizontal waves.) On substituting we see that the imaginary part of $n^2+i\dfrac{n}{H_{00}}$ is $i\left(2\nu n_i+\dfrac{\nu}{H_{00}}\right)$ so that making this zero requires $n_i=-\dfrac{1}{2H_{00}}$. Thus the real part of $n^2+i\dfrac{n}{H_{00}}$ becomes $\nu^2+\dfrac{1}{4H_{00}{}^2}$.

The upshot of all this is that effectively we are seeking solutions of the form

$$\psi'=\left[A\exp\frac{z^*}{2H_{00}}\right]\exp i(\sigma t+\lambda x+\mu y+\nu z^*)$$

Eq 9

and for this to be a solution we require

$$\nu^2=\frac{N^2}{f_0^2}\left[\frac{\beta}{u_0+c}-(\lambda^2+\mu^2)\right]-\frac{1}{4H_{00}{}^2},$$

Eq 10

where $c = -\sigma / \lambda$ is the phase speed of the wave towards the east.

In view of what follows we note that we could always have begun by seeking solutions of the form of Eq 9 whether or not we expect ν to be wholly real, but without the benefit of the previous analysis it would seem a perverse thing to do. For now our position should be that we are expecting ν to be wholly real, but it may turn out not to be.

Stationary baroclinic Rossby waves

As for barotropic Rossby waves, we shall study the case of waves driven by the undulations in the underlying surface. Thus we are interested in stationary waves with $c = 0$. By considering the sign of the various terms in Eq 10 we conclude that wave-like solutions (i.e. with $\nu^2 > 0$, so that ν is real) are only possible when $\frac{\beta}{u_0}$ is sufficiently positive, which requires that u_0 is positive but smaller than a certain critical value, u_c say, given by

$$u_c = \beta\left[\lambda^2 + \mu^2 + \left(\frac{f_0}{2NH}\right)^2\right]^{-1}$$

Eq 11

Thus we can distinguish two cases, dependent on whether ν is wholly real or wholly imaginary. It turns out, as we shall show below, that when it is wholly real, the waves propagate in the vertical and in the other case they do not. These are known as the *propagating* case and the *trapped* (a.k.a. *evanescent*) case respectively.

Let us consider in more detail what determines which case obtains. Our assumption is that the waves are driven by flow over the underlying surface. Let us suppose first that this takes the form of a simple wavy underlying surface with height above mean sea level proportional to $\exp i(\lambda_m x + \mu_m y)$ where the subscript m is to show that these wavenumbers give the scale of the mountain. As the air is forced to blow over this surface, a pattern of vertical velocity must be set up which has the same horizontal scales, so that it has the same horizontal variation proportional to $\exp i(\lambda_m x + \mu_m y)$ but possibly with a different phase. It follows from the thermodynamic equation that $\frac{\partial \psi}{\partial z^*}$ has this same variation, so that $(\lambda, \mu) = (\lambda_m, \mu_m)$. Thus (λ, μ) is actually set by the scale of the mountain. This in turn means that u_c is set by the underlying mountain. If the basic wind is westerly and exceeds this value, the waves will be trapped, as they will for easterly winds. If the wind is westerly but less than the critical value set by the scale of the topography the waves will be propagating. More realistic lower boundaries will have more complicated mountains, which can be represented as a Fourier spectrum of the simple waves. Each member of the spectrum will have its own (λ_m, μ_m) and hence its own u_c. In westerly winds if that value of u_c is larger than the basic wind, then the corresponding wave will be propagating, otherwise it will be trapped. From the expression for u_c we can see that large waves (small wavenumbers) need bigger values of the basic wind to trap them. Thus large waves are more likely to be propagating than small ones are.

Now we need to look at what the properties of the propagating and trapped cases are in more detail.

Propagating case

This case has $0 < u_0 < u_c$ and hence ν is real. The magnitude of the solution $|\psi'|$ is proportional to $\exp \frac{z^*}{2H_{00}} = \left(\frac{p_0}{p}\right)^{\frac{1}{2}}$. It follows that the magnitude of $p(u'^2 + v'^2)$ is constant with z^*. When the corresponding analysis is performed in height (as opposed to z^*) co-ordinates then the magnitude of $\rho(u'^2 + v'^2)$ is found to be constant with height, so that the kinetic energy density of the perturbed part of the flow is constant with height.

The phase of the waves is constant on the surface $(\lambda x + \mu y + \nu z^*) = const.$ These intersect the latitudinal planes $y = const$ in the lines $(\lambda x + \nu z^*) = const.$ These phase lines in the latitudinal plane slope towards the west with increasing height if λ and ν have the same sign and to the east otherwise.

We can show (see tutorial questions) that when the phase lines slope westwards the group velocity is directed upwards, while in the other case it is directed downwards. Since we expect the waves to be driven by the interaction of the air and the underlying topography, the energy source must be at the surface and not at great heights. Thus the phase lines which slope westward with height correspond to reality, while the waves which slope eastwards with increasing height are unrealistic.

It is also possible to show (again see the tutorial questions) that for a fixed latitude and height the waves which slope westwards with increasing height have poleward travelling air which is warmer than equatorwards travelling air, so these waves transport sensible heat towards the poles.

Trapped case

This case has either $u_0 < 0$ or $u_c < u_0$ and hence ν is wholly imaginary and of the form $\nu = \pm i\nu_i$ with ν_i real and positive.

There are two cases to consider, according to which of the plus and minus signs we take. It is obvious that in both cases the phase lines are vertical, as the solution now looks like

$$\psi' = \left[A \exp\left(\frac{1}{2H} \mp \nu_i\right) z^*\right] \exp i(\sigma t + \lambda x + \mu y)$$

Eq 12

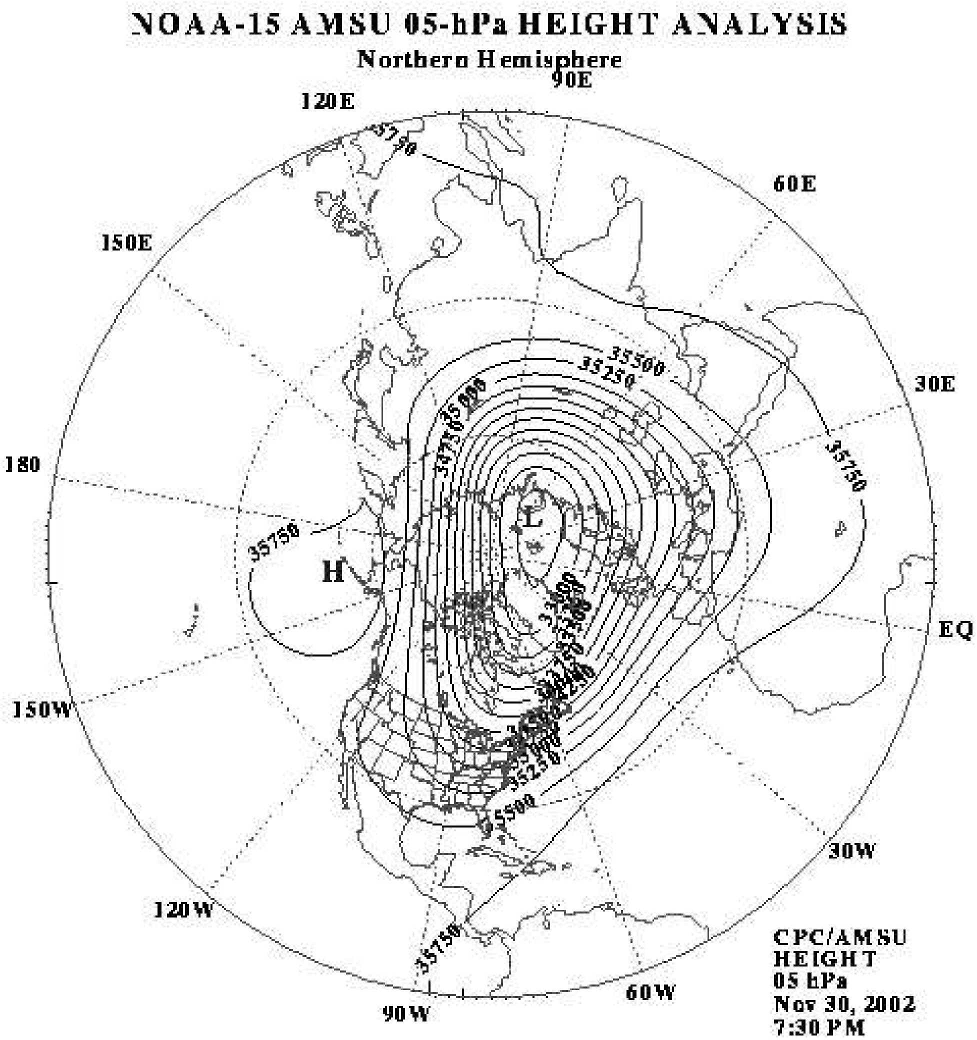
The variation in the vertical is not wavelike, but is either an exponential growth or decay. The kinetic energy density $p\left(u'^2 + v'^2\right)$ is easily seen to increase upwards if we take the positive sign in Eq 12. This is inconsistent with our idea that the waves are related to the bottom boundary. Hence we must take the minus sign to give

$$\psi' = \left[A \exp\left(\frac{1}{2H_{00}} - \nu_i \right) z^* \right] \exp i(\sigma t + \lambda x + \mu y).$$

Note that because there is no wavelike variation in the vertical, there is no propagation of energy in the vertical.

We usually have $\nu_i > \dfrac{1}{2H_{00}}$, so usually the magnitude of ψ' decreases exponentially upwards, as do the perturbation velocities and temperatures, but under some circumstances they may grow upwards (but obviously slowly enough that the kinetic energy density still decreases upwards).

There is a small part of this chapter still to be written. This will summarise the overall findings above, and show how it is consistent with the observed behaviour of the atmosphere as exemplified in the maps below.



NOAA-17 AMSU 05-hPa HEIGHT ANALYSIS

Southern Hemisphere