Atmospheric Dynamics

Lect.13: The Quasi-geostrophic Equations

Stephan Matthiesen

Institute for Atmospheric and Environmental Sciences

School of GeoSciences, Edinburgh University

Crew Building room 218

Scope of Lecture

- f-plane and β-plane
- Systematic use of geostrophic equation
- The Stream Function
- Quasi-geostrophic potential vorticity

f-plane and β-plane approximations

- Both use tangent plane approx.
- For f-plane, f=constant=f_o
- For β-plane, f is made a linear function of y

$$f = f_0 + \beta\left(y - y_0\right) \qquad \beta = \left(\frac{\partial f}{a\partial\phi}\right)_0 = \frac{2\Omega\cos\phi_0}{a}$$

for $\phi_0 = 45^o N \qquad \beta = \dfrac{10.3\times10^{-5}s^{-1}}{6371km} = 1.62\times10^{-11}m^{-1}s^{-1}$

Stream function

$$u_g \equiv -\frac{1}{f}\frac{\partial \varphi}{\partial y}, v_g \equiv -\frac{1}{f}\frac{\partial \varphi}{\partial x}$$

$$f \approx f_0$$

$$u_g \equiv -\frac{1}{f_0}\frac{\partial \varphi}{\partial y}, v_g \equiv -\frac{1}{f_0}\frac{\partial \varphi}{\partial x}$$

define

$$\psi \equiv \frac{\varphi}{f_0}$$

$$u_g = -\frac{\partial \psi}{\partial y} \quad v_g = +\frac{\partial \psi}{\partial x} \longrightarrow \mathbf{v}_g = \mathbf{k} \wedge \nabla_p \psi$$

ψ is the *streamfunction*

Streamfunction (cont)

$$\zeta_{rel} \sim \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla_p^2 \psi$$

$$div\mathbf{v}_g = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

Quasi-geostrophic vorticity equation

We had

$$\left(\frac{D}{Dt}\right)_p (\zeta_{rel} + f) = (\zeta_{rel} + f)\frac{\partial \varpi}{\partial p}$$

$$\left(\frac{D}{Dt}\right)_p \zeta_{rel} + v\beta = (\zeta_{rel} + f)\frac{\partial \varpi}{\partial p}$$

typically

$$\zeta_{rel} / f = U / Lf = R_o$$

$$\left(\frac{D}{Dt}\right)_p \zeta_{rel} + v_g\beta = f_0\frac{\partial \varpi}{\partial p}$$

Streamfunction and temperature

$$\frac{\partial \psi}{\partial p} = \frac{\partial}{\partial p}\left(\frac{\varphi}{f_0}\right) = -\frac{1}{f_0 \rho}$$

$$\frac{\partial \psi}{\partial p} = -\frac{R}{f_0 p} T$$

The thermodynamic equation

We shall treat adiabatic motion

$$\frac{D\theta}{Dt} = 0$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)_p \theta + \varpi\frac{\partial \theta}{\partial p} = 0$$

$$\theta = Tp^{-\kappa}{p_0}^{\kappa}$$

$$p^{-\kappa}{p_0}^{\kappa}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)_p T + \varpi\frac{\partial \theta}{\partial p} = 0$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)_p T + \varpi\frac{T}{\theta}\frac{\partial \theta}{\partial p} = 0$$

Thermodynamic eq. (cont)

Write $\frac{T}{\theta}\frac{\partial\theta}{\partial p} = -\Gamma$ (measure of static stability)

replace Γ with its area average, Γ_0

and replace horizontal velocity components with geostrophic approximations:-

$$\left(\frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}\right)_p \frac{\partial\psi}{\partial p} + \frac{\Gamma_0 R}{f_0 p}\varpi = 0$$

The Quasi-geostrophic equations

The complete evolution of the flow is governed by the two equations

$$\left(\frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}\right)_p \nabla_p{}^2\psi+v_g\beta=f_0\frac{\partial\varpi}{\partial p}$$

$$\left(\frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}\right)_p \frac{\partial\psi}{\partial p}+\frac{\Gamma_0 R}{f_0 p}\varpi=0$$

Quasi-geostrophic potential vorticity

Eliminating ω from previous two gives

$$\left(\frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}\right)_p q_g = 0$$

where

$$q_g \equiv \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)_p \psi + f_0 + \beta y + \frac{f_0^{\ 2}}{R}\frac{\partial}{\partial p}\left(\frac{p}{\Gamma_0}\frac{\partial \psi}{\partial p}\right)$$

$$= \left(\xi_{rel}\right)_g + f + \frac{f_0^{\ 2}}{R}\frac{\partial}{\partial p}\left(\frac{p}{\Gamma_0}\frac{\partial \psi}{\partial p}\right)$$

Quasi-geostrophic approximation

Primitive eq'ns with selective approximations

- Horizontal winds replaced by geostrophic winds
- Basic-state static stability as function of vertical coordinate only

Appropriate for extra-tropical, synoptic-scale systems

Also called: pseudogeostrophic or geostrophic approximation