Chapter 13: The Quasi-Geostrophic Equations

In our search for explanation of the behaviour of atmospheric phenomena, we will need to try to write down all the equations which govern atmospheric behaviour and try to solve them simultaneously. In their raw form, the equations prove too complex for analytical treatment, and we have either to solve the full set using computer techniques or we have to somehow simplify the equations to the point where analytical solutions are possible and hope that the solutions to the simplified equations still contain useful insights to the real behaviour. In this course we are following the latter strategy. We will develop these with p as the vertical co-ordinate.

We have already seen one level of simplification when we adopted the tangent-plane approximation and used rectangular Cartesian co-ordinates instead of the full spherical co-ordinates. In this chapter we develop a set of equations which has proved tractable, by systematically applying the geostrophic approximation.

The f-plane and β-plane

We shall continue to adopt the tangent-plane approximation, but the next step in our development requires us to decide how to treat the Coriolis parameter, f . This is a trigonometrical function of latitude, but latitude on the tangent plane is no longer defined in the quite the same way.

The f-plane

The simplest treatment is simply to replace every occurrence of f by a constant value f_0. The resulting set of equations are said to follow *the f-plane approximation*. The "plane" is to remind us that we are using the tangent plane, and the "f" is to remind us that we have constant Coriolis parameter (and by the same token the vertical component of the Earth's rotation is treated as constant everywhere in this approximation). We shall make use of this approximation in a later chapter when studying cyclone development.

The β-plane

The f-plane approximation is too restrictive for some purposes. The next simple level of treatment is to allow f to vary in the north-south direction but to make it only a linear function. Thus we write $f = f_0 + \beta(y - y_0)$, where $\beta = \left(\frac{\partial f}{a\partial\phi}\right)_0 = \frac{2\Omega\cos\phi_0}{a}$ and y_0 is the value of y where $f = f_0$. Choosing $\phi_0 = 45^o N$ gives

$$\beta = \frac{10.3\times10^{-5}s^{-1}}{6371km} = 1.62\times10^{-11}m^{-1}s^{-1}$$

The stream function

With p as vertical component, the equations for the components of the geostrophic wind (u_g, v_g) are defined as $u_g \equiv -\frac{1}{f}\frac{\partial \varphi}{\partial y}, v_g \equiv -\frac{1}{f}\frac{\partial \varphi}{\partial x}$, where ϕ is the geopotential.

Using that approximation and the further approximation that the scales are such that $f_0 >> \beta(y - y_0)$ the expressions for the geostrophic wind components become $u_g \equiv -\frac{1}{f_0}\frac{\partial \varphi}{\partial y}, v_g \equiv -\frac{1}{f_0}\frac{\partial \varphi}{\partial x}$. It proves useful to define $\psi \equiv \frac{\varphi}{f_0}$.

This gives $u_g = -\frac{\partial \psi}{\partial y}$, $v_g = +\frac{\partial \psi}{\partial x}$, so that

$$\boxed{\mathbf{v}_g = \mathbf{k} \wedge \nabla_p \psi}$$

Eq 1

ψ is a *streamfunction*. The (geostrophic) flow is parallel to lines of constant ψ and its strength is proportional to the spacing of iso-lines of ψ.

$$\boxed{\zeta_{rel} \sim \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla_p^2 \psi}$$

Eq 2

$$\boxed{div\mathbf{v}_g = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y \partial x} = 0}$$

Eq 3

Note that Eq 1 to Eq 3 are all appropriate for both the f-plane and the β-plane.

The vorticity equation

Previously we had the vorticity equation in the form

$$\left(\frac{D}{Dt}\right)_p (\zeta_{rel} + f) = (\zeta_{rel} + f)\frac{\partial \varpi}{\partial p}$$

or

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)_p \zeta_{rel} + v\frac{\partial f}{\partial y} = -(\zeta_{rel} + f)\frac{\partial \varpi}{\partial p}.$$

Now the typical magnitude of $\zeta_{rel} / f = U / Lf = R_o$, so that on the right of the equation the relative vorticity can be neglected compared to the Coriolis parameter to the order of the Rossby number. Moreover in all the terms on the left the wind can be replaced by the geostrophic wind, to the same order of approximation. The result is

$$\boxed{\left(\frac{D}{Dt}\right)_g \nabla_p{}^2\psi + v_g\beta = f_0\frac{\partial\varpi}{\partial p}}$$

Eq 4

with $\left(\frac{D}{Dt}\right)_g \equiv \left(\frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}\right)_p$

The subscript g on the material derivative is to remind us that it is to be evaluated using the geostrophic approximation and the subscript p is to remind us to hold pressure constant while doing the differentiation.

Note that the left hand side of Eq 4 can be written entirely in terms of the streamfunction, so the equation contains two dependent variables, ψ and ϖ. Thus for a full specification of the physical system, we will need another equation in these two variables. Now Eq 4 can be regarded as a version of the momentum equation which we have massaged somewhat, so the other equation will need to come from a different physical principle. It does not take much thought to realise that the physical statement which we have not employed in producing Eq 4 is the first law of thermodynamics. We can regard the momentum equation as governing the movement of the centre of mass of the cloud of molecules which make up the air particles, while the first law of thermodynamics keeps track of the energy of the movement of the molecules about their centre of mass (i.e. the *internal energy* of the particles). In writing down that law in the form appropriate to large-scale motion it is useful to note the following relationship.

Relation between streamfunction and temperature

In an earlier chapter, we noted that vertical gradients of geostrophic wind are related to horizontal gradients of temperature, we might expect there to be a simple relationship between the vertical derivative of the streamfunction and temperature. Indeed there is, and it is readily found, for

$\frac{\partial\psi}{\partial p} = \frac{\partial}{\partial p}\left(\frac{\varphi}{f_0}\right) = -\frac{1}{f_0\rho}$ by the hydrostatic equation (remember $\delta\varphi \equiv g\partial z$). Thus

$$\boxed{\frac{\partial\psi}{\partial p} = -\frac{R}{f_0 p}T}$$

Eq 5

The thermodynamic equation.

For the purposes of this course, we shall assume that the motion is adiabatic. Thus as an air parcel moves about, its entropy and hence its potential temperature remains constant. This approximation is acceptable for many days in the stratosphere, but only for a few days in the troposphere. Locally (where latent heat is released) it can be quite a poor approximation, but the areas of precipitation in large-scale flow are relatively small. Thus our thermodynamic equation, without any approximation apart from that of negligible heat supply to the air particles, is $\frac{D\theta}{Dt} = 0$, or

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)_p \theta + \varpi\frac{\partial \theta}{\partial p} = 0$$

Now since $\theta = Tp^{-\kappa}{p_0}^{\kappa}$, we shall have

$$p^{-\kappa}{p_0}^{\kappa}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)_p T + \varpi\frac{\partial \theta}{\partial p} = 0$$

which is easily rewritten as

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)_p T + \varpi\frac{T}{\theta}\frac{\partial \theta}{\partial p} = 0.$$

Eq 6

Now we introduce our approximations for synoptic scale flow, i.e. we will treat flows for which the Rossby Number is small and the horizontal velocity components are replaced by the geostrophic versions.

We shall also introduce a small approximation to the final term on the left hand side. First write $\frac{T}{\theta}\frac{\partial \theta}{\partial p} = -\Gamma$, so that Γ is a measure of static stability. In the strongly stratified situations of synoptic scale motion Γ is a function of pressure, but its fractional variations in the horizontal are only a few percent, whereas the fractional variations of ϖ in the horizontal are large, changing sign between its maximum and minimum values. The essence of Eq 6 is therefore captured if we replace Γ by its area average (the average being taken at constant pressure). If we denote this area average by Γ_0 and substitute for T from Eq 5, then Eq 6 becomes

$$\boxed{\left(\frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}\right)_p \frac{\partial \psi}{\partial p} + \frac{\Gamma_0 R}{f_0 p}\varpi = 0}$$

Eq 7

The pair of equations Eq 4 and Eq 7 consist entirely of terms which can be written in terms of the two dependent variables ψ and ϖ. As we have two equations in two unknowns, they completely determine the evolution of the flow, and if their distribution is known at a given instant, the future evolution can also be found (subject to suitable boundary conditions). They are known as the *quasi-geostrophic equations* and the set of approximations which produced them is known as the *quasi-geostrophic approximation*.

Notice that we have not said that he wind is exactly geostrophic, all we have done is said that it is approximately geostrophic, and given that it is so (i.e. that the Rossby number is small) we have found the relationship between the first order variables when terms of order the Rossby number are neglected. If we chose to, we could deduce equations which would allow us to calculate those smaller terms. For instance as the stream function evolves in time, it is clear that there are accelerations in this flow. We could use the geostrophic wind to estimate the accelerations and hence we could estimate the ageostrophic components (just as we did in an earlier chapter).

There are two important new equations which can be deduced from Eq 4 and Eq 7, one by eliminating ϖ between them, the other by eliminating the local time derivatives of ψ.

Quasi-geostrophic potential vorticity

ϖ may be eliminated by writing it as the subject of Eq 7 and substituting into Eq 4. It is left as an exercise for the student to derive the result, which is

$$\left(\frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}\right)_p q_g=0$$

Eq 8

where

$$\begin{aligned} q_g &\equiv \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi + f_0 + \beta y + \frac{f_0^{\,2}}{R}\frac{\partial}{\partial p}\left(\frac{p}{\Gamma_0}\frac{\partial \psi}{\partial p}\right) \\ &= \xi_{rel} + f + \frac{f_0^{\,2}}{R}\frac{\partial}{\partial p}\left(\frac{p}{\Gamma_0}\frac{\partial \psi}{\partial p}\right) \end{aligned}$$

Eq 9

Eq 8 tells us that the quantity q_g is conserved following the qeostrophic flow. From Eq 9 we see that this is the absolute vorticity, modified by adding to it a term involving the vertical temperature gradients. Hence we may identify q_g as a sort of potential vorticity which is unchanging for an observer moving with the geostrophic flow. Note that it has the form of a modified Laplacian of the stream function.

The omega equation

If we eliminate $\frac{\partial \psi}{\partial t}$ from Eq 4 and Eq 7 by forming $\frac{\partial}{\partial p}$ of Eq 4 and $\nabla_p^{\,2}$ of Eq 7, then we obtain (after multiplying by $\frac{f_0 p}{\Gamma_0 R}$),

$$\nabla_p^{\,2}\varpi + \frac{f_0^{\,2}p}{\Gamma_0 R}\frac{\partial^2\varpi}{\partial p^2} = \frac{f_0 p}{\Gamma_0 R}\left\{\frac{\partial}{\partial p}\left[(\mathbf{v}_g.\nabla_p)\nabla^2\psi - v_g\beta\right] - \nabla_p^{\,2}\left[\mathbf{v}_g.\nabla_p\frac{\partial\psi}{\partial p}\right]\right\}$$

The left hand side is a modified 3-D Laplacian of the vertical velocity ϖ, while the right hand side comprises two terms, one the vertical gradient of vorticity advection, the other the horizontal (2-D) Laplacian of temperature advection. The right hand side can be found from the instantaneous distributions of wind (or streamfunction), enabling the vertical velocity to be found by inverting the 3-D Laplacian subject to suitable boundary conditions. Suitable boundary conditions are $\varpi = 0$ at the surface and at the top of the atmosphere and at the edges of the region of interest.