Chapter 11: Relation between vorticity, divergence and the vertical velocity

The divergence equation

In chapter 3 we used a simple version of the continuity equation. Here we develop it further, partly because it will give useful relationship between horizontal divergence and vertical velocities and partly because it is one of the fundamental equations which we will need when we attempt to solve the equations of motion to deduce the properties of various meteorological phenomena.

The fundamental idea is that the mass of a marked lump of fluid stays constant, so that for a small lump of volume $\delta\tau$, say, and density ρ, the product of volume and density remains constant, i.e. $\frac{D}{Dt}(\rho\delta\tau)=0$. If the mass is constant, so is its volume, so that we also have $\frac{D}{Dt}\ln(\rho\delta\tau)=0$. This can be re-written

$$\frac{1}{\rho}\frac{D\rho}{Dt}+\frac{1}{\delta\tau}\frac{D\delta\tau}{Dt}=0.$$

Eq 1

Now we remarked in chapter 11 that $\frac{1}{\delta\tau}\frac{D\delta\tau}{Dt}=div\mathbf{v}$. (It is the full 3-D divergence which appears in this expression.) We did not prove this result, but you were invited to prove the 2-D analogue of it as one of the problems on that chapter, and the generalisation to 3-D is straightforward. Putting that result into Eq 1 and expanding the derivative gives

$$\frac{1}{\rho}\left\{\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}\right\}+div\mathbf{v}=0$$

Eq 2

Other ways of writing this are

$$\boxed{\frac{D}{Dt}\rho+\rho div\mathbf{v}=0}$$

Eq 3

and

$$\boxed{\frac{\partial\rho}{\partial t}+div(\rho\mathbf{v})=0}$$

Eq 4

Eq 3 and Eq 4 are the continuity equation. Being written in vector form those equations make no assumption about the co-ordinate system. There is an approximation to these equations which we can make. To demonstrate this, we will write the equation in our tangent plane rectangular co-ordinate system. Another way of grouping the terms is

$$\left\{\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}\right\}+\rho div_h\mathbf{v}_h+\rho\frac{\partial w}{\partial z}=0$$

or

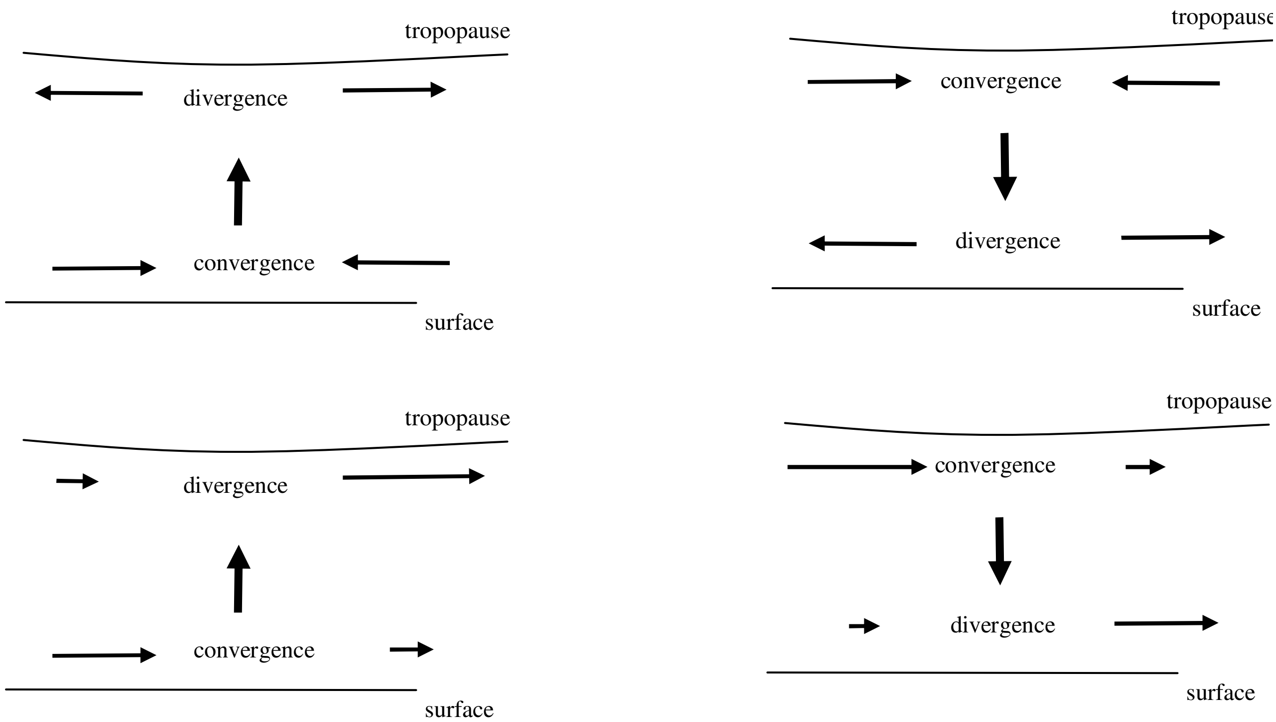
$$\left\{\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}\right\}+\rho div_h\mathbf{v}_h+\frac{\partial(w\rho)}{\partial z}=0.$$

The term in curly brackets in the last equation can be shown to be small compared with the other terms, so to good approximation the continuity equation is

$$\boxed{\rho div_h\mathbf{v}_h+\frac{\partial(w\rho)}{\partial z}=0}$$

Eq 5

We see that there is a relationship between the vertical velocity and the horizontal divergence. Vertical velocities usually have their largest magnitude in the middle of the troposphere. This is not surprising, as the surface must mean that vertical velocities are zero actually at the surface. In addition the high static stability of the stratosphere suppresses vertical velocities in the vicinity of the tropopause. Thus the largest vertical velocities occur in the middle of the troposphere. If there is upward motion in the middle troposphere we shall have $\frac{\partial(w\rho)}{\partial z}>0$ in the lower troposphere and $\frac{\partial(w\rho)}{\partial z}<0$ in the upper troposphere. It follows that in this case $div_h\mathbf{v}_h<0$ in the lower troposphere and $div_h\mathbf{v}_h>0$ in the upper troposphere. That is to say that middle level ascending motion has (horizontal) convergence below it and divergence above it, while descending motion in middle levels has divergence below it and convergence above it.



The sketches above show some possible configurations of vertical and horizontal velocities and hence horizontal convergences and divergences.

Magnitude of the vertical velocity

We can use the approximate continuity equation (Eq 5) to estimate typical values of the vertical velocity. Expanding that equation we obtain

$$\frac{\partial w}{\partial z}+\frac{w}{\rho}\frac{\partial \rho}{\partial z}=-div_h\mathbf{v}$$

Now if we use U,W,L,H for typical orders of magnitude for, respectively, the horizontal, and vertical velocities and horizontal and vertical distances, then the terms on the left-hand side of the equation are of order $\frac{W}{H}$ We have seen in the previous chapter that the right hand side is of order $R_o\frac{U}{L}$. Hence we must have that $\frac{W}{H}\sim R_o\frac{U}{L}$ or

$$W\sim R_oU\frac{H}{L}.$$

The ratio $\frac{H}{L}$ for mid-latitude systems is $\sim\frac{10km}{1000km}=\frac{1}{100}$, so we might have expected that vertical velocities were about one hundredth of horizontal velocities, but our analysis shows that we must multiply by an additional factor of the Rossby number (namely 1/10 in for the mid-latitude synoptic scale). Thus the vertical velocities are typically 1/1000 time the horizontal ones. This gives a typical large-scale mid-latitude vertical velocity of 1cm/sec or 1km per day.

A consequence of these magnitudes is that the vertical advection terms in the material derivative are small. The material derivative is $\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}$. The second and third terms are of order of magnitude U/L and we have previously shown that the timescales are such that the first term is of similar magnitude. However the final term is of magnitude $\frac{W}{H}=\frac{R_oUH}{HL}=R_o\frac{U}{L}$ which is an order of magnitude smaller and may be ignored to order of the Rossby number, or in symbols:-

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+O(R_o)$$

The vorticity equation

An important relation between the vorticity and divergence (and hence vertical velocities) emerges when we derive the rate of change of vorticity. To obtain this we start from the equations of motion, in which we have neglected the vertical advection term.

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Take $\frac{\partial}{\partial y}$ of the first equation from $\frac{\partial}{\partial x}$ of the second, giving

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}+f\frac{\partial u}{\partial x}+f\frac{\partial v}{\partial y}+v\frac{\partial f}{\partial y}$$
$$=-\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y}+\frac{1}{\rho}\frac{\partial^2 p}{\partial y\partial x}$$

We have ignored the terms in the horizontal gradients of density, as being small compared to the retained terms. The terms on the right cancel and the remainder of the equation can be re-arranged to give

$$\left(\frac{D}{Dt}\right)_h(\zeta_{rel}+f)=-(\zeta_{rel}+f)div_h\mathbf{v}_h$$
$$\left(\frac{D}{Dt}\right)_h(\zeta_{abs})=-(\zeta_{abs})div_h\mathbf{v}$$

Eq 6

This is known as the *vorticity equation*. We have added a subscript h to the material derivative to remind ourselves that only the horizontal terms matter. Sometimes this is described as “the rate of change following the horizontal motion” for obvious reasons.

Absolute vorticity is usually positive in the northern hemisphere (and negative in the southern). According to this equation, the magnitude of the absolute vorticity is decreased by horizontal divergence and increased by horizontal convergence. This is a consequence of the conservation of angular momentum. Consider a lump of air. Convergence decreases the horizontally projected area of the lump. Hence it decreases its moment of inertia. To maintain a constant angular momentum the lump has to spin faster. The opposite occurs for the case of horizontal divergence.

Using the continuity equation in the form of Eq 5 to eliminate the divergence from Eq 6 gives

$$\left(\frac{D}{Dt}\right)_h(\zeta_{abs})=\frac{\zeta_{abs}}{\rho}\frac{\partial(\rho w)}{\partial z}$$

The term on the right is positive if ρw is increasing upwards. The dominant effect is whether w increases upwards or not. If it does it means that the air columns are stretching. Stretching air columns lead to an increase in the magnitude of the absolute vorticity, while shrinking air columns lead to a decrease in the magnitude of the absolute vorticity.

Vorticity and Divergence equations in pressure co-ordinates

We shall find it useful to use pressure co-ordinates in later developments. It can be shown, by analysis which will need to be taken on trust in this course, that for hydrostatic atmospheres, the continuity equation in pressure co-ordinates becomes

$$\boxed{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \varpi}{\partial p}=0}$$

Eq 7

or

$$\boxed{div_p \mathbf{v}_h+\frac{\partial \varpi}{\partial p}=0}$$

The three-dimensional divergence is zero in pressure co-ordinates. Note that there are no time derivatives in this equation.. Since the hydrostatic equation implies $\delta p=-\rho g \delta z$, we might expect $\varpi=\frac{Dp}{Dt}\sim-\rho g\frac{Dz}{Dt}=-\rho g w$. The similarity between Eq 7 and the approximate form of the continuity equation in the previous chapter is thus apparent. However, it turns out that, while that equation is only approximate, Eq 7 is exact to the extent that the hydrostatic equation holds.

The vorticity equation in pressure co-ordinates becomes

$$\boxed{\left(\frac{D}{Dt}\right)_p(\zeta_{abs})=\zeta_{abs}\frac{\partial \varpi}{\partial p}}$$

Eq 8 a

$$\text{a}\boxed{\left(\frac{D}{Dt}\right)_p(\zeta_{abs})=-\zeta_{abs}div_p \mathbf{v}_h}$$

Eq 9 b