Atmospheric Dynamics Lect.10: Vorticity and Divergence

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Scope of Lecture

- Definition of vorticity and divergence
- Physical interpretation
- Approximate non-divergence of synoptic-scale motions

Definition of vorticity

3-D vorticity is defined as the curl of the wind velocity

$$curl\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Since vertical scales and velocities are small we have

$$curl\mathbf{v} \sim -\mathbf{i}\left(\frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Magnitude of components

Horizontal components ~ $(10ms^{-1})/(10km) \sim 10^{-3}s^{-1}$

Vertical component ~ $(10ms^{-1})/(1000km) \sim 10^{-5}s^{-1}$

So horizontal components are about 100 times the vertical component.

Relative vorticity

As velocity components are relative to Earth a subscript *rel* is applied.

Vertical component is of most meteorological relevance despite size.

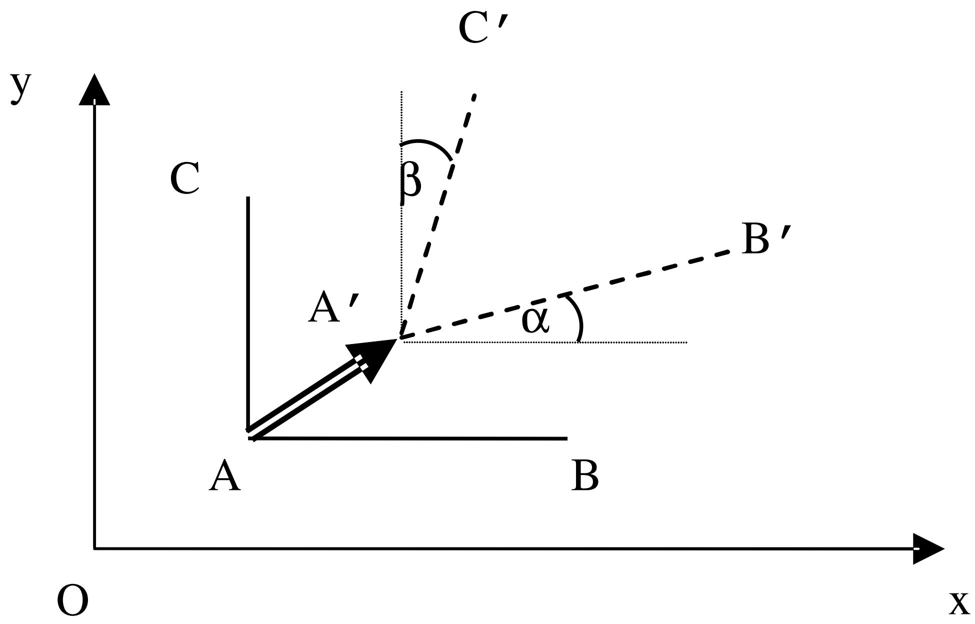
Often just called *the vorticity*

or *the relative vorticity*

$$\zeta_{rel} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

Physical interpretation of vorticity

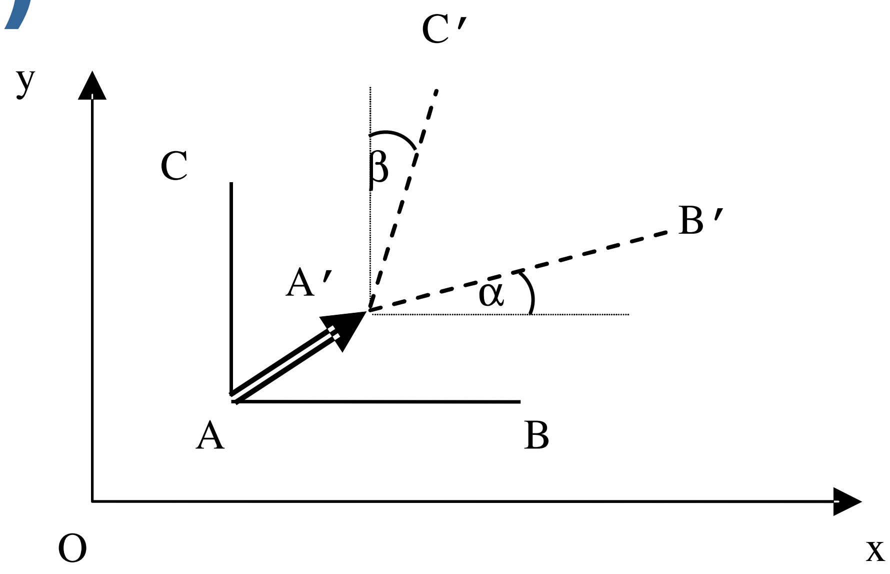
Marked lines of fluid AB, AC move to A′B′ and A′C′ in time δt



A, B, C are $\left(x_o, y_o\right)$ $\left(x_o + \delta x, y_o\right)$ $\left(x_o, y_o + \delta y\right)$ respectively

A′ is easily seen to be $\left(x_o + u_o \delta t, y_o + v_o \delta t\right)$

Physical interpretation of vorticity (cont)



A′ is

$$\left(x_o + u_o \delta t, y_o + v_o \delta t\right)$$

B′ is

$$\left(x_o + \delta x + \left(u_o + \frac{\partial u_o}{\partial x}\delta x\right)\delta t, y_o + \left(v_o + \frac{\partial v_o}{\partial x}\delta x\right)\delta t\right)$$

x-comp. of A′B′ $= \delta x + O\left(\delta x \delta t\right)$

y-comp. of A′B′ $= \frac{\partial v}{\partial x}\delta x \delta t + O\left(\delta x^2 \delta t\right)$

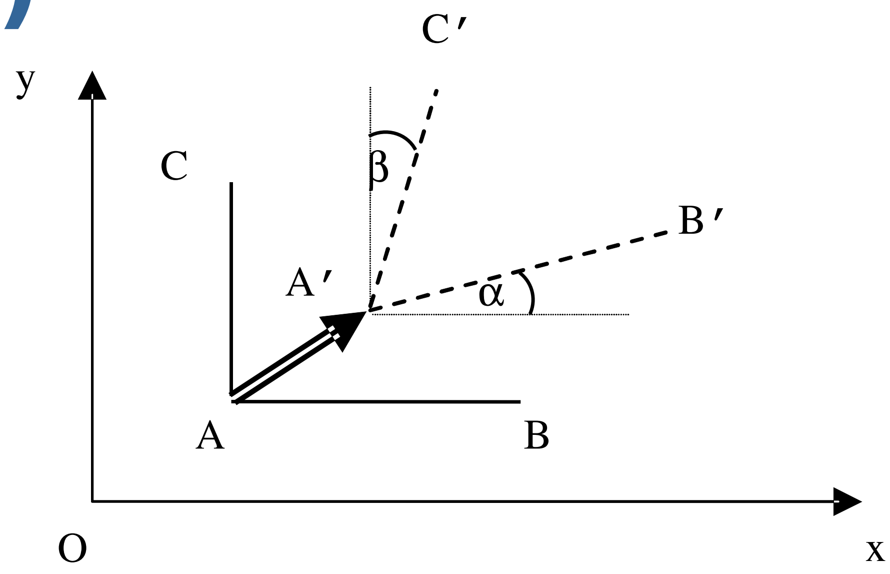
--> $\alpha = \frac{\partial v}{\partial x}\delta t$

Physical interpretation of vorticity (cont)

$$\alpha = \frac{\partial v}{\partial x}\delta t$$

likewise

$$\beta = \frac{\partial u}{\partial y}\delta t$$



Average rotation rate $= \frac{1}{2}\left(\alpha - \beta\right) / \delta t$

Vorticity=twice rate of rotation.

Absolute vorticity

To get absolute vorticity add on twice earth's rotation about vertical

$$\zeta_{abs} = \zeta_{rel} + 2 \times \Omega \sin\phi$$

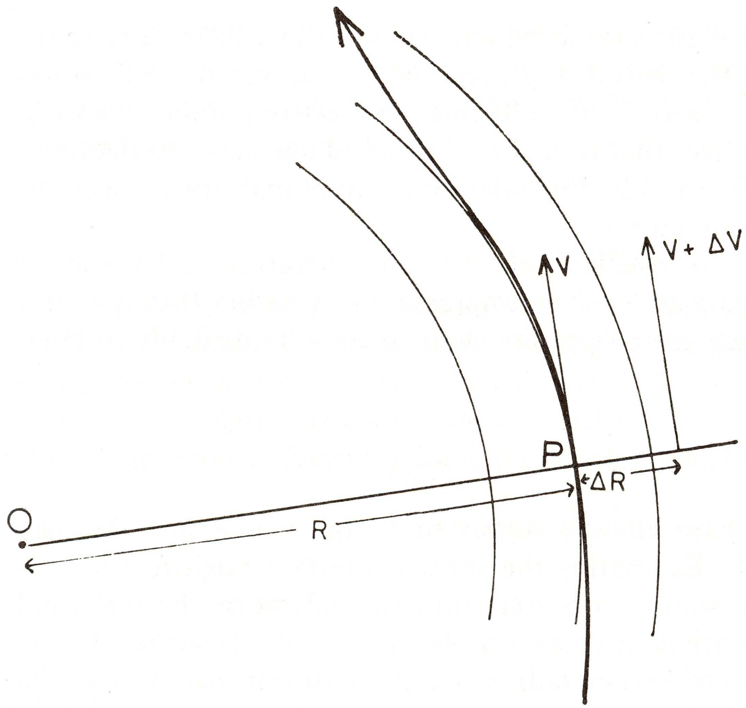
$$\zeta_{abs} = \zeta_{rel} + f$$

Alternative form

- From definition vorticity must be independent of the axes
- Take local axes Ox'y' with y' axis in direction of flow and x' perpendicular to it. So for anticlockwise flow (i.e. cyclonic in N. Hemis.) is radially outwards away from local centre of curvatur

$$\varsigma = \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} = \frac{\partial V}{\partial R} + \frac{V}{R}$$

where R is the radius of curvature



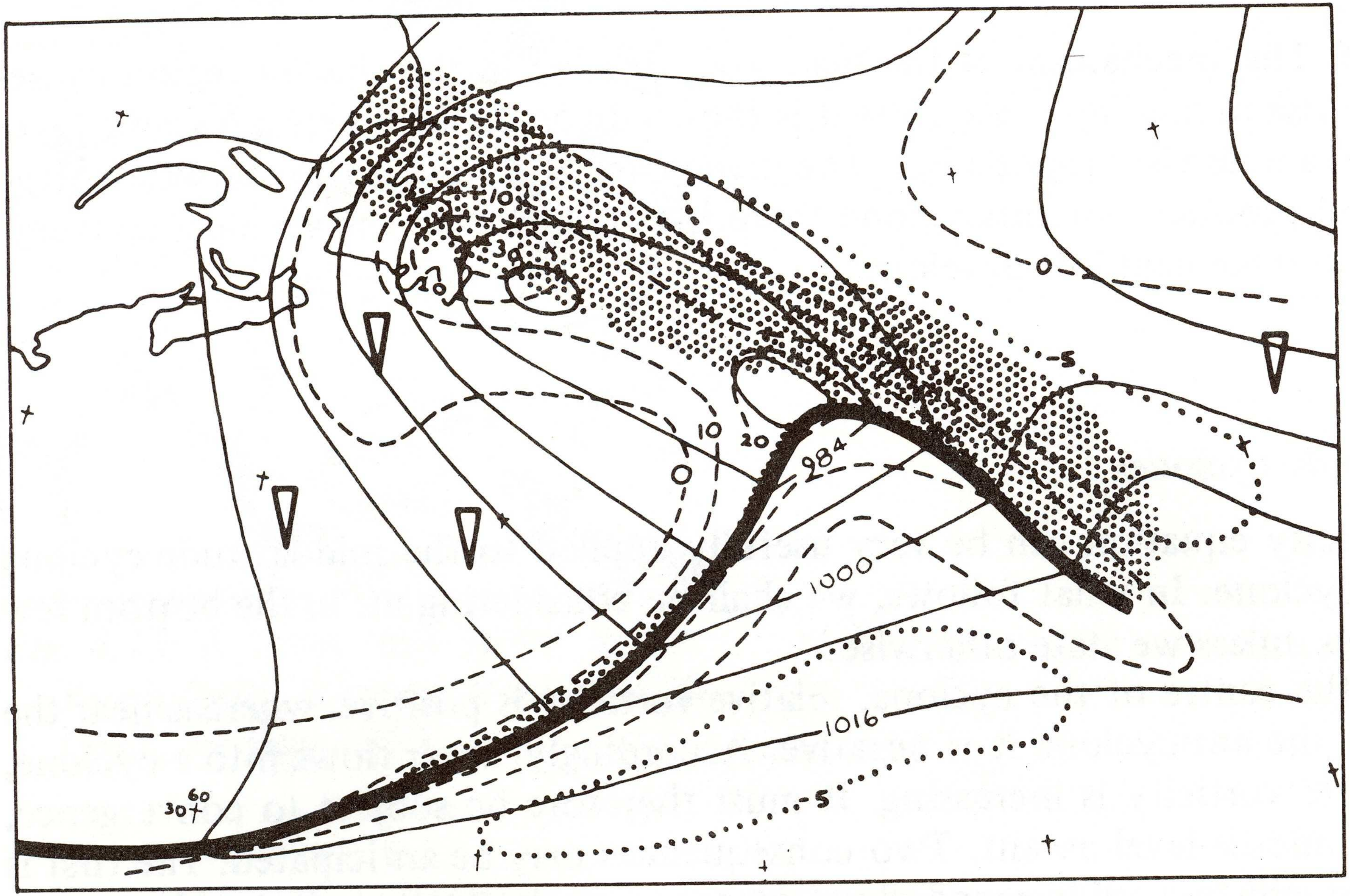
Vorticity

$$\zeta = \frac{\partial V}{\partial R} + \frac{V}{R}$$

Net vertical vorticity is sum of two parts:

- Shear vorticity
- Curvature vorticity

Surface vorticity (estimated from geostrophic wind)



Dashed lines are lines of equal rel. vorticity: units of $10^{-5}s^{-1}$

Divergence

3D divergence defined:-

$$div\mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

2D version most useful

$$div_h \mathbf{v}_h = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

We can show that for element δA,

$$div_h \mathbf{v}_h = \frac{1}{\delta A} \frac{D\delta A}{Dt}$$

Parcels expand for +ve divergence

contract for –ve divergence

= convergence

Divergence is small

Consider div of geostrophic wind

$$div_h \mathbf{v}_g = \left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} \right) = -\frac{\partial}{\partial x}\left(\frac{1}{\rho f} \frac{\partial p}{\partial y} \right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho f} \frac{\partial p}{\partial x} \right)$$

$$= \frac{1}{\rho f}\left\{ -\frac{\partial^2 p}{\partial x \partial y} + \frac{\partial^2 p}{\partial y \partial x} \right\} - \frac{1}{\rho f^2} \frac{\partial p}{\partial x} \frac{\partial f}{\partial y}$$

hence $$div_h v_g = 0 - \frac{v_g}{f} \frac{\partial f}{\partial y}$$

now $$\frac{1}{f} \frac{\partial f}{\partial y} = \frac{1}{2\Omega \sin\phi} \frac{1}{a} \frac{\partial}{\partial \phi} 2\Omega \sin\phi = \frac{1}{a} \cot\phi$$

Smallness of divergence (cont)

At 45°N

$$\frac{1}{a}\cot\phi = \frac{1}{6.4\times10^{6}m} = 10^{-7}m^{-1}$$

$$div_h v_g \sim 10ms^{-1}10^{-7}m^{-1} = 10^{-6}s^{-1}$$

Thus the divergence is about 10% of the magnitude of the terms which comprise it. These are therefore approximately equal and of opposite sign

As $\mathbf{v}_g$ is an $O(R_o)$ approx to $\mathbf{v}$, real wind is quasi-nondivergent too