# Chapter 10: Vorticity and divergence

In this chapter we look at two important derivatives of the wind field, *vorticity* and *divergence*.

## *Formal definition of vorticity*

The vorticity plays an important role in fluid dynamics generally. It is defined as the curl of the wind, namely $curl\mathbf{v}$.

$$curl\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

As the velocities in this expression are measured relative to a frame of reference fixed in the earth, this quantity is known as the *relative vorticity*.

Since in general the horizontal velocities are much larger than vertical velocities, and vertical scales are much smaller than horizontal scales, in the x and y components of that expression we can neglect the terms in the vertical velocity, giving

$$curl\mathbf{v} \sim -\mathbf{i}\left(\frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

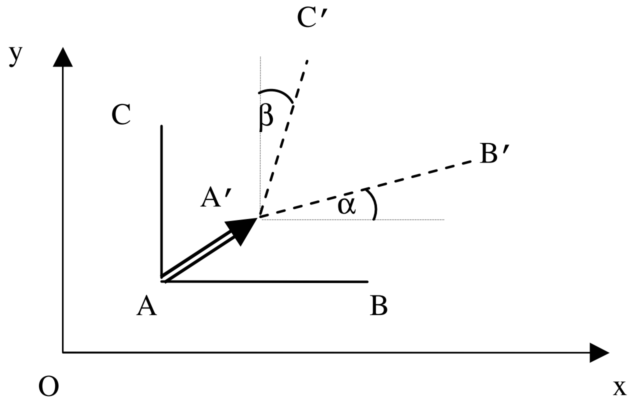
The first two terms on the right have typical magnitude $(10ms^{-1})/(10km) \sim 10^{-3}s^{-1}$, while the third has typical magnitude $(10ms^{-1})/(1000km) \sim 10^{-5}s^{-1}$. Clearly the horizontal components are typically about 100 times as large as the vertical component. However, in large-scale meteorology it is the study of the vertical component which proves most fruitful, and we shall concentrate on that from now on and not pay much attention to the horizontal components. In meteorology the term relative vorticity is often used simply to denote this vertical component. The context will normally make it clear if the full vector vorticity or just the vertical component is intended. We denote the vertical component of the relative vorticity by $\zeta_{rel}$. Thus

$$\boxed{\zeta_{rel} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}$$

As we have seen above, a typical magnitude for this in extra-tropical flow is $10^{-5}s^{-1}$.

## *Physical interpretation*

We can show that $\zeta_{rel}$ is related to the rotation of the fluid particles about a vertical axis. To see this, consider two horizontal marked lines of fluid (it may help to imagine them marked with smoke) and follow their motion. We label their position at time t as AB and AC. Initially AB is parallel to the x-axis and AC is parallel to the y-axis, and we will assign them lengths $\delta x$ and $\delta y$ respectively.




In time $\delta t$ AB moves to A′B′ and AC moves to A′C′. In the general case, when the lines reach the new positions they are no longer parallel to the axes nor horizontal.

If A is $(x_o, y_o)$ then B is $(x_o + \delta x, y_o)$, and C is $(x_o, y_o + \delta y)$.

A′ is readily seen to be the point $(x_o + u_o\delta t, y_o + v_o\delta t)$, where the subscript zeros on the velocity components are to remind us that they are evaluated at $(x_o, y_o)$. B′ can be calculated in a similar way, except for the difference that the velocity components are different at B than they are at A. This can be accounted for by a simple first order Taylor expansion, giving the horizontal co-ordinates of B′ as

$$\left( x_o + \delta x + \left( u_o + \frac{\partial u_o}{\partial x}\delta x \right)\delta t, y_o + \left( v_o + \frac{\partial v_o}{\partial x}\delta x \right)\delta t \right)$$

Thus the x-component of A′B′ $= \delta x + O(\delta x \delta t)$

and the y-component of A′B′ $= \frac{\partial v}{\partial x}\delta x \delta t + O(\delta x^2 \delta t)$

Hence the angle α in the sketch is given by $\alpha = \frac{\partial v}{\partial x}\delta t$ and we see that $\frac{\partial v}{\partial x}$ is the rate at which AB rotates *anti-clockwise*.

A similar argument shows that $\frac{\partial u}{\partial y}$ is the rate at which AC rotates *clockwise*.

If we consider circular lumps of fluid, for which AB and AC contribute equally, we see that the average rate of anticlockwise rotation is $\frac{1}{2}\left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$, the minus sign in the second term being needed because the convention is anticlockwise is positive.

Hence $\zeta_{rel}$ is twice the instantaneous rate of rotation of circular fluid particles about a vertical axis.

We often need to concern ourselves with the vorticity as seen from inertial space. This can be found by adding to the relative vorticity the extra effects of the Earth's rotation, i.e. we add twice the Earth's rate of rotation about a vertical axis. The vertical component of vorticity as seen from inertial space is denoted by. Thus

$$\zeta_{abs} = \zeta_{rel} + 2\times\Omega\sin\phi \qquad \text{or} \qquad \zeta_{abs} = \zeta_{rel} + f$$

Note that in middle latitudes the Coriolis parameter has magnitude $10^{-4}s^{-1}$, so the relative vorticity is about an order of magnitude smaller than the absolute vorticity. You should have little problem convincing yourself that the typical ratio of $\zeta_{rel}$ to $f$ is the Rossby number.

## *Vorticity in terms of curvature and shear*

This bit still to be written, but the gist of it can be seen from the slides associated with this lecture.

## *Vorticity in a real situation*

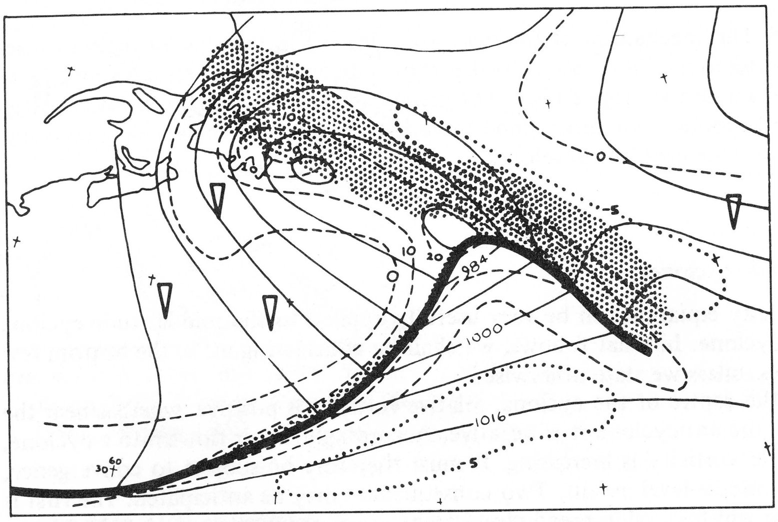



**Figure 1:** The solid lines are isobars, the dashed lines a isopleths of constant positive relative vorticity and the dotted lines are isopleths of negative relative vorticity, labelled in units of 10-5s-1. Shading indicates rain area and triangles an area of showers.

Figure 1 above shows the vorticity associated with the surface weather chart in a middle latitude cyclone part way through its life cycle. Over much of the chart the relative vorticity is smaller than the Coriolis parameter, consistent with the argument above, but at the low pressure centre and associated with the fronts there is large positive relative vorticity, amounting to a few times the Coriolis parameter. Actually the values have been calculated on the basis of the geostrophic approximation, and as that overestimates the wind in regions of cyclonic curvature, the true vorticity will be somewhat less than the maxima shown here.

## *Divergence*

Next we introduce the other important derivative of the windfield, namely the *divergence*. Again in many fluid-dynamical applications a three dimensional quantity is most relevant, namely $div\mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$. However in the meteorological context the horizontal part of this[1] is particularly useful, namely

$$div_h \mathbf{v}_h = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

**Eq 1**

If $\delta\tau$ is an element of volume, and $\delta A$ is an element of horizontal area, then we can show that $div\mathbf{v} = \frac{1}{\delta\tau}\frac{D\delta\tau}{Dt}$ and $div_h \mathbf{v}_h = \frac{1}{\delta A}\frac{D\delta A}{Dt}$. Thus $div_h \mathbf{v}_h$ is the fractional rate of increase of an element of area of a marked fluid particle. $div_h \mathbf{v}_h$ should, strictly speaking, be called the *horizontal divergence*, but often the "horizontal" is dropped as 3-D divergence is not widely used in meteorology.

Divergence is *positive* when the area is *increasing*.
Divergence is *negative* when the area is *decreasing*.

Negative divergence is often referred to as *convergence*.

### Smallness of the horizontal divergence.

The expression for the divergence contains terms which superficially are similar to those in the definition of vorticity, namely horizontal gradients of the horizontal wind components. However it turns out that the divergence is typically an order of magnitude smaller than the vorticity, as there is an approximate cancellation between the two terms in the definition (Eq 1). To see this, first consider the divergence of the geostrophic wind:-

$$\begin{aligned} div_h \mathbf{v}_g &= \left( \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} \right) = -\frac{\partial}{\partial x}\left( \frac{1}{\rho f}\frac{\partial p}{\partial y} \right) + \frac{\partial}{\partial y}\left( \frac{1}{\rho f}\frac{\partial p}{\partial x} \right) \\ &= \frac{1}{\rho f}\left\{ -\frac{\partial^2 p}{\partial x \partial y} + \frac{\partial^2 p}{\partial y \partial x} \right\} - \frac{1}{\rho f^2}\frac{\partial p}{\partial x}\frac{\partial f}{\partial y} \end{aligned}$$

[1] Of course the divergence is a scalar quantity; this talk of 2 or 3 dimensions refers to the wind components used

We have neglected some horizontal gradients in density as being small. Hence we have deduced that

$$div_h \mathbf{v}_g = 0 - \frac{v_g}{f}\frac{\partial f}{\partial y}.$$

Now $\frac{1}{f}\frac{\partial f}{\partial y} = \frac{1}{2\Omega \sin\phi}\frac{1}{a}\frac{\partial}{\partial \phi} 2\Omega \sin\phi = \frac{1}{a}\cot\phi$.

At 45$^o$N $\frac{1}{a}\cot\phi = \frac{1}{6.4\times 10^6 m} = 10^{-7} m^{-1}$, so that

$$div_h \mathbf{v}_g \sim 10ms^{-1} 10^{-7} m^{-1} = 10^{-6} ms^{-1}.$$

Now we have already seen that both $\frac{\partial u_g}{\partial x}$ and $\frac{\partial v_g}{\partial y}$ have typical magnitudes of $10^{-5} ms^{-1}$. Thus the sum of these terms is an order of magnitude less than either of them taken individually. In other words $\frac{\partial u_g}{\partial x}$ and $\frac{\partial v_g}{\partial y}$ are approximately equal and opposite to each other. If they were exactly equal and opposite (so that the sum was zero) then the geostrophic flow would be said to be *non-divergent*. What we have shown is that the geostrophic wind is approximately non-divergent.

It follows that the true wind must also be approximately horizontally non-divergent because $div_h \mathbf{v}_h = div_h \mathbf{v}_g + div_h \mathbf{v}_a$ and $|\mathbf{v}_a| \sim \frac{1}{R_o}|\mathbf{v}_g|$, giving $div \mathbf{v}_a \sim R_o \frac{U}{L} \sim \frac{1}{10} 10^{-5} s^{-1}$. Thus $div_h \mathbf{v}_h \sim 10^{-6} s^{-1}$, so the two terms cancel approximately in this quantity also.

## Estimation from wind measurements

Note that this cancellation makes the horizontal divergence much harder to estimate from observations than the relative vorticity, because a ten percent error (say) in the horizontal wind components will translate into a ten percent error in relative vorticity, but into a hundred percent error in horizontal divergence, the difference in behaviour arising from the cancellation in the latter case.