Chapter 8: Wind distribution in the boundary layer: The Ekmann Spiral

In this chapter we consider how friction changes the wind in the layer near to the surface. The layer affected by friction with the surface varies in depth with location and occasion according to the state of the convection, the strength of the pressure gradients and whether there is large-scale ascent in the middle levels of the troposphere. Sometimes it is called the *planetary boundary layer*. Often there is quite a sharp transition between the boundary layer and the layer of *free atmosphere* above it in which friction is unimportant. A typical depth of the planetary boundary layer is 1 km, but it can be just a few hundred meters, and sometimes it could be considered to fill the troposphere. It is beyond our scope to try to treat all the possible varieties. Here we shall simply give a simple treatment of what might be considered the basic case.

We shall assume that the wind in the boundary layer is steady and does not vary in the horizontal. A better way of stating this is that the effects of variations in time and of horizontal gradients are small compared with the effects of the vertical momentum transports. This feels plausible considering the boundary layer is perhaps only a kilometre deep, while the horizontal variations happen over several hundreds of kilometres, and we have seen that the time variations are compatible with the spatial scales through the typical velocities. Of course it is dangerous to simply assume that we can neglect these time variations and horizontal variations, so a sensible procedure once we have found the solution would be to check up on the size of terms in the equations which we left out compared with those we retained. This assumption of steadiness and no horizontal variations means that there will be no relative acceleration, so the equation at the end of the last chapter simplifies to

$$f\mathbf{k}\wedge\overline{\mathbf{v}}_h = -\frac{1}{\rho}\nabla_h p + \frac{1}{\rho}\frac{\partial}{\partial z}\left(\rho K\frac{\partial\overline{\mathbf{v}}_h}{\partial z}\right).$$

A further assumption is that the variations in density in the boundary layer are small compared with the variations in velocity. This approximation is somewhat cruder than the previous one, but it allows us to reach an analytical solution. Applying the approximation the equation simplifies to

$$f\mathbf{k}\wedge\overline{\mathbf{v}}_h = -\frac{1}{\rho}\nabla_h p + \frac{\partial}{\partial z}\left(K\frac{\partial\overline{\mathbf{v}}_h}{\partial z}\right).$$

The simplest case in which we can solve this equation is when the eddy viscosity, K, is independent of height. This reduces the equation to

$$f\mathbf{k}\wedge\overline{\mathbf{v}}_h = -\frac{1}{\rho}\nabla_h p + K\left(\frac{\partial^2\overline{\mathbf{v}}_h}{\partial z^2}\right).$$

Finally we shall assume that the geostrophic wind is independent of height. To exploit that, expand the wind into geostrophic and ageostrophic components $\overline{\mathbf{v}}_h = \mathbf{v}_g + \mathbf{v}_a$.

$$f\mathbf{k} \wedge (\mathbf{v}_g + \mathbf{v}_a) = -\frac{1}{\rho}\nabla_h p + K\left(\frac{\partial^2(\mathbf{v}_g + \mathbf{v}_a)}{\partial z^2}\right)$$

$$f\mathbf{k} \wedge \mathbf{v}_a = +K\left(\frac{\partial^2 \mathbf{v}_a}{\partial z^2}\right).$$

This equation can be solved for suitable boundary conditions. As height is the only independent variable in this equation, we can replace the partial derivatives with total derivatives, We shall write it in component form as the coupled pair of equations:-

$$K\frac{d^2 u_a}{dz^2} + fv_a = 0$$
$$K\frac{d^2 v_a}{dz^2} - fu_a = 0$$

These are simultaneous equations. One way of solving them is by brute force elimination of one variable to leave a 4th order equation for the other. However clever trick which allows us to remain dealing with 2nd order equations is to define a complex variable U, say[1], by $U = u_a + iv_a$, where $i + = -1$. If we multiply the second equation by i and add it to the first we get

$$K\frac{d^2 u_a}{dz^2} + iK\frac{d^2 v_a}{dz^2} + fv_a - ifu_a = 0$$

or

$$K\frac{d^2 U}{dz^2} - ifU = 0.$$

This has solution $U = A\exp(+i\beta z) + B\exp(-i\beta z)$, where A and B are constants to be determined by the boundary conditions, and $\beta = \sqrt{\frac{-if}{K}} = (i-1)\sqrt{\frac{f}{2K}}$.

One boundary condition comes from considering what happens at great heights. U is effectively the ageostrophic wind, and the only thing which causes ageostrophy in this problem is the friction at the surface. Hence we expect that $|U \to 0|$ as $z \to \infty$. This will require that $B = 0$, because the term involving B can be written $B\exp\left(z\sqrt{\frac{f}{2K}}\right)\exp\left(iz\sqrt{\frac{f}{2K}}\right)$. The second exponential always has magnitude 1, but the first one grows exponentially as z increases. Thus we need $B = 0$ to confine our solution to near to the surface. Leaving:-

[1] Note that this is a different use of U from our previous use of it as a typical windspeed.

$$U = A\exp(+i\beta z)$$

If we write $\alpha \equiv \sqrt{\frac{f}{2K}}$, then the previous expression becomes

$U = A\exp(-\alpha z)\exp(-i\alpha z)$, which can be written in the form $re^{i\theta}$ with $r = A\exp(-\alpha z)$ and $\theta = -\alpha z$. This makes it easy to see that the magnitude of U decreases exponentially as z increases and the direction of U in the complex plane rotates uniformly clockwise as z increases. Such a curve is known as a geometric spiral.

The sketch in Figure 1 shows the locus of U in the Argand diagram. The real and imaginary axes are shown as dotted lines. O is the origin of these axes. P shows the general position of U for height z. As z increases the line OP rotates clockwise and P moves along the curve, closer and closer to O. S shows the position for $z = 0$, where $U = A$. For the moment A is arbitrarily chosen, but, as we have already stated, we need to determine it from the boundary conditions.

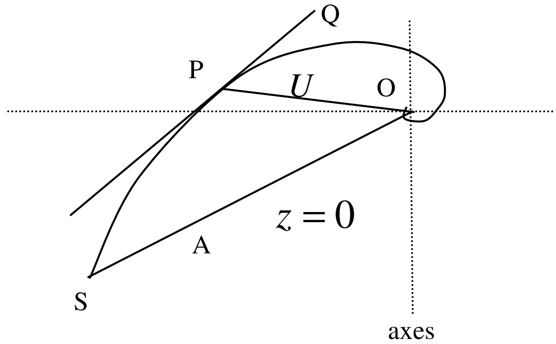

Figure 1: Ageostrophic wind

PQ is the tangent to the curve. We note that $\frac{dU}{dz} = i\beta A\exp(i\beta z) = i\beta U$. This means that the tangent PQ is rotated from OP by the same angle as $i\beta$ is rotated from the real axis (since when you multiply to complex numbers together the product is obtained by multiplying their magnitudes and adding their arguments.

Now $i\beta = -(i+1)$ so this has argument $\frac{5\pi}{4}$. It follows that $\angle OPQ$ is $\frac{\pi}{4} = 45^o$. Note that this angle is independent of z. For this reason the geometric spiral is sometimes also called the *equi-angular spiral*, as it is the locus of a point moving so that the direction of movement is at a constant angle to its displacement vector from the origin. As the angle is constant, the acute angle made between the tangent at S and the line SO is also 45^o.

Various boundary conditions are possible according to the degree of approximation considered acceptable in representing the physics. We shall use the "no-slip" boundary condition. This assumes that the surface roughness brings the air which is in contact with the surface to a standstill, so that the total wind is exactly zero at the surface there. If we denote the geostrophic wind in the complex plane by $U_g = u_g + iv_g$, then the total wind is the sum of U_g and U. At the ground this sum must be zero, which requires that $A = -U_g$.

Since the total wind is obtained by adding U_g to the ageostrophic wind previously found, we get the situation sketched in Figure 2. This has some similarities with the Figure 1 but the origin of the axes is now at S and the curve OPS now represents the locus of total wind as height changes. The line OP has not been drawn in, but it would represent the ageostrophic wind as previously. SP shows the total wind at the general height z. The line SO represents the geostrophic wind. As z increases, P moves round the curve in the direction of O, approaching O asymptotically. As z decreases, P moves round the curve nearer to S, that is, as we get nearer and nearer to the ground the wind gets less and less. The tangent SR at the surface is shown. Just above the surface the wind is in the direction of that tangent. We showed when discussing Figure 1 that $\angle OSR$ is 45°. Hence the surface wind blows at an angle of 45° to the geostrophic wind, and is directed towards the low pressure side

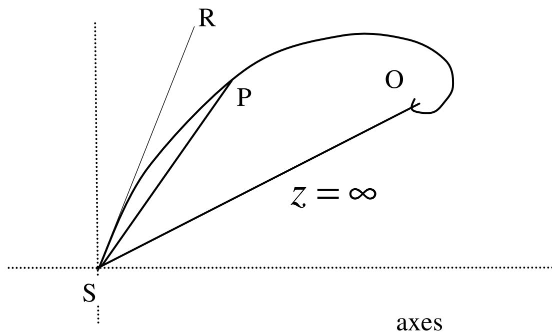

Figure 2: The total wind

As the variations expressed in our solution extend to infinity, albeit getting always closer to the geostrophic wind, there is no obvious top to the boundary layer in our treatment. Accordingly it is usual to consider the top to be where the total wind is first in the same direction as the geostrophic wind. This is readily seen to happen at a height z_t, say, where the ageostrophic wind is first lined up with the geostrophic wind and in the same direction. This is in the opposite direction to A (which is the ageostrophic wind at the surface where the total wind is zero. For the ageostrophic wind to have rotated π radians from A we need $z_t\alpha = \pi$, so that

$$z_t = \pi\sqrt{\frac{2K}{f}}.$$

The eddy viscosity is difficult to measure directly, but the wind can be measured much more readily, and we typically find, by studying the wind direction, that z_t is about 1km. This allows us to estimate typical values of the eddy viscosity, K, as 5 m^2s^{-1}.