Chapter : Friction

Molecular friction (viscosity) is unimportant in large-scale flow, as the velocity shears are too small. However there are almost always systematic motion systems of a scale smaller than the scale of interest which have an effect analogous to viscosity. For instance near the surface the presence of obstacles such as trees or buildings or uneven ground produces swirls and eddies on the scale of centimetres to many meters. Moreover the thermally induced motions associated with cumulus clouds produce systematic motions with the scale of kilometres. These motions can transfer momentum from one part of the fluid to another by interchanging air between fast and slowly moving layers of air. These small-scale motion systems last only for a few seconds or several minutes, so one way to consider their effects is to regard the flow as the combination of a slowly varying part (the part on the scale we wish to study) and a rapidly fluctuating part (the small-scale "eddies" or swirls) superimposed upon it. (See anemometer trace in the slides pertaining to this lecture.)

We can isolate the slowly varying part by applying a filter comprising the time average, which we will denote by an overbar. Thus for a variable s we define a local time mean $\bar{s}$ centred on the time t_o by

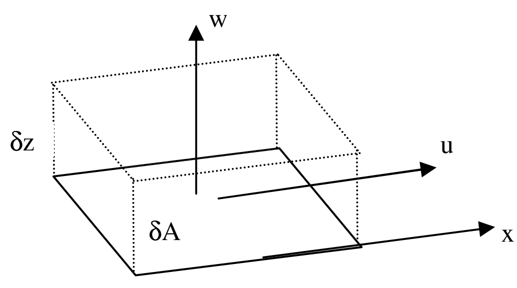
$$\bar{s} = \frac{1}{2Q}\int_{t_o-Q}^{t_o+Q} s\,dt\ ,$$

where Q is the time interval which is long compared with the eddy motion, but short compared with the time-scale of the large-scale flow. Q may be chosen in the range 15 to 30 minutes for instance. We shall denote the departure from the mean by a dash, and call it the "eddy" part of the flow. Thus the eddy part of s is given by

$$s' \equiv s - \bar{s}$$

We note a pair of relations deducible from this definition which we will need later, namely $\bar{\bar{s}} = \bar{s}$ and $\overline{s'} = 0$.

We can now consider how fluxes of momentum are carried by the eddy and mean parts of the flow. Specifically we will begin by deriving the vertical flux of x-momentum.



Consider a small horizontal area δA. In time δt the volume passing through this area is $w\delta t\delta A$, so the mass passing through is $\rho w\delta t\delta A$ and the x-momentum passing through is $u\rho w\delta t\delta A$. Thus the flux upwards of x-momentum per unit time, per unit area through a horizontal area is $u\rho w$.

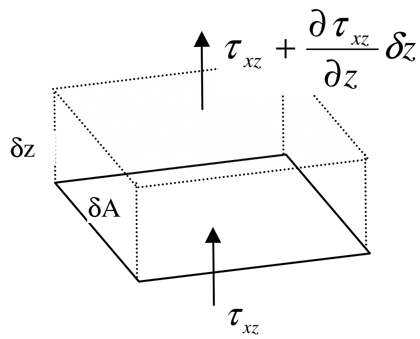
What we have just worked out was the instantaneous flux. The time mean is $\overline{u\rho w}$. Writing $u = \overline{u} + u'$ etc this becomes $\overline{(\overline{\rho} + \rho')(\overline{u} + u')(\overline{w} + w')}$. Now it turns out that the fractional variations in density are much smaller than those in the velocity components, as temperature and pressure vary by less than a percent or so, whereas the vertical velocity may change by 100 percent and the horizontal velocity by many 10s of a percent. Thus the variations of density can be neglected compared with those in the other quantities, and the vertical flux of x-momentum can be written[1]

$$\overline{(\overline{\rho})(\overline{u} + u')(\overline{w} + w')} = \rho\overline{(\overline{u} + u')(\overline{w} + w')} = \rho\overline{(\overline{u}\overline{w} + u'\overline{w} + \overline{u}w' + u'w')}$$
$$= \rho(\overline{u}.\overline{w}) + \rho(\overline{u'w'}).$$

The first term on the right hand side consists entirely of terms from the large-scale, slowly varying flow, whereas the second term arises from the eddies, and hence is called the *eddy flux of x-momentum*. The first term represents the momentum fluxes which are already expressed in the equations of motion which we deduced and used in earlier chapters, while this eddy flux of x-momentum represents a new phenomenon which is not yet included in those equations and which therefore requires an amendment to them. There are analogous expressions for the vertical flux of y-momentum.[2] We can regard this eddy flux of momentum as a frictional force or *eddy viscous force* produced by *eddy viscosity*. It can be regarded as a force (per unit area) in the x-direction exerted on the fluid above the surface by the fluid above. Clearly it will be positive if the u' and w' are positively correlated, that is if upward moving air is travelling (in the x-direction) faster than the average flow and downward moving air is travelling slower than the average.

We may write $\tau_{xz} = \rho(\overline{u'w'})$ for the upward flux of x-momentum, and $\tau_{yz} = \rho(\overline{v'w'})$ for the upward flux of y-momentum.

Now consider the flux of x-momentum into and out of a box with horizontal area δA , lying between the heights of z and z+δz. The x-momentum entering the box through the bottom surface in time δt is $(\tau_{xz})\delta A.\delta t$ while the x-momentum leaving the box through the top surface is $\left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial z}\delta z\right)\delta A.\delta t$.



Since the mass of the box is $\rho\delta A\,\delta z$, the change in velocity in time δt corresponding to the change in the x-momentum in the box produced by the difference of these two fluxes is

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[1] Note that the dots which appear in the next few expressions denote an ordinary multiplication. If I miss them out, then the overbars join up when I wish to show them being separate.

[2] And even of the eastward flux of y momentum, but the fact that the large scale flow has horizontal dimensions which are very large compared with the vertical ones makes the effects of this small compared with the vertical fluxes, so that we do not need to consider them here.

$$\left((\tau_{xz})\delta A.\delta t - \left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial z}\delta z \right)\delta A.\delta t \right) \div (\rho \delta A\, \delta z) = -\frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z}\delta t \,.$$

Hence the acceleration produced by the eddy viscous forces in the x-direction is

$$-\frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z}$$

Thus the vector acceleration can be written $-\frac{1}{\rho}\frac{\partial \boldsymbol{\tau}}{\partial z}$, where we have written $(\tau_{xz}, \tau_{yz}) \equiv \boldsymbol{\tau}$ for the vertical flux of (horizontal vector) momentum.

We see therefore that we can incorporate the eddy viscosity into the equations of motion by adding this term to the equations to give

$$\boxed{\frac{D\mathbf{v}_h}{Dt} + fk \wedge \mathbf{v}_h = -\frac{1}{\rho}\nabla_h p - \frac{1}{\rho}\frac{\partial \tau}{\partial z}}$$

The quantities the left and the pressure gradient term are understood to be written in terms of the mean flow quantities. Hence they should strictly be written with overbars, but in practice no difficulty arises f rom missing them out, and it makes for simpler writing and typography.

The problem now is to find the eddy momentum fluxes. Our fervent hope is that these are determined by the properties of the mean flow alone, because then, provided we can write expressions for the eddy fluxes in terms of the mean-flow variables, we will end up with equations which only contain mean-flow variables. By analogy with molecular friction, we hope that the vertical flux of momentum is proportional to the shear in the vertical of the horizontal wind. If $\overline{u}$ increases upwards we expect τ_{xz} to be directed downwards (as interchanging fluid between the fast-moving upper layers and the slower moving lower layers will bring x-momentum downwards) and to be proportional to $\frac{\partial \overline{u}}{\partial z}$. Moreover the denser the fluid, the more momentum is transferred. Thus we might expect a relationship like

$$\tau_{xz} = -\rho K \frac{\partial \overline{u}}{\partial z},$$

where K is a constant.

This relationship works surprisingly well, especially in conditions of neutral stability. K is found to vary with distance from the surface close to the surface, and to depend on how rough the surface is. It also depends on the stability, as might be anticipated, since in stable conditions buoyancy forces will inhibit the vertical exchange of air parcels. Given this variability it may be wondered if we have made any advance by adopting this formulation, but providing the nature of the variability can be found, it has allowed us to obtain an expression for the effects of the eddies which can be evaluated entirely on the basis of the mean flow quantities.

K is called the *coefficient of eddy viscosity* by analogy with the (kinematic) coefficient of molecular viscosity. \sometimes it is called the eddy diffusion coefficient, as it is often the case that heat and moisture are diffused in a similar way to momentum and with the same constant. It is about 6 orders of magnitude larger than the coefficient of molecular viscosity. We shall show in the next chapter one way of estimating it, which gives typical values of $\sim 5m^2s^{-1}$.

We shall have the analogous relation $\tau_{yz} = -\rho K \frac{\partial \overline{v}}{\partial z}$ and the combined vector form, $\boldsymbol{\tau} = -\rho K \frac{\partial \overline{\mathbf{v}}_h}{\partial z}$, leading to

$$\boxed{\frac{D\mathbf{v}_h}{Dt} + fk \wedge \mathbf{v}_h = -\frac{1}{\rho}\nabla_h p + \frac{1}{\rho}\frac{\partial}{\partial z}\left(\rho K \frac{\partial \mathbf{v}_h}{\partial z}\right)}.$$

All the quantities in this equation are now on the resolvable scale, but we have dropped the overbar (and will continue to do so from now on) to simplify the typography.