Chapter 6: Other vertical co-ordinates and the thermal wind

Other vertical co-ordinates

There are advantages in having other quantities besides z as the vertical co-ordinate. Several have been proposed for use in meteorology. One idea is to use material surfaces in the fluid. Suppose for instance that we could introduce into the atmosphere at a given instance and at each place a certain amount of smoke with a mixing ratio which depended only on its height at that instance. We could (for a short time at least) then subsequently locate each air parcel by quoting its x-coordinate, its y-coordinate and the mixing ratio of the smoke (assuming it was not changed by diffusion). That seems a fairly complicated thing to do. Indeed it is, but the use of a coordinate system which moves with the flow in that way turns out to have several advantages in some contexts[1].

If the motion is known to be statically stable and the processes are nearly adiabatic, then potential temperature turns out to be a good choice for vertical co-ordiante, because vertical velocities can be very hard to estimate in conventional height co-ordinates, but in the θ system "vertical" velocities are very simply related to the heating rate, and this can be much more readily calculated. Indeed, if the motion is exactly adiabatic, motion in the θ system is two-dimensional, which gives us certain conceptual advantages too.

Other co-ordinates which have been proposed include pressure, and various functions of pressure, such as ln(p), and pressure divided by the pressure at the surface. This latter co-ordinate is particularly useful because the surface looks simple in this system, being simply where the coordinate has the value 1. This has some advantages when writing computer codes to predict weather and climate.

Use of pressure as vertical co-ordinate

We will only pursue one system of vertical co-ordinates here, namely the one in which pressure is used as the vertical co-ordinate. In this system instead of the co-ordinates being (x, y, z), they are (x, y, p) In the new system z is a *dependent* variable. The idea of using pressure as a vertical co-ordinate is most useful when the atmosphere is close to being in hydrostatic equilibrium. This is not strictly necessary, but some of the principal advantages are lost in the case when hydrostatic equilibrium does not obtain. What is absolutely necessary is that pressure is a monotonic function of height, otherwise the mathematical transformations cannot be performed. Clearly this condition is satisfied in the case of hydrostatic equilibrium, because the pressure at any height is just the weight per unit area of the atmosphere above that height and this always decreases with height.

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[1] Co-ordinate systems which follow the motion of the fluid are said to be *Lagrangian*, whereas static systems are *Eulerian*, after two different ways of developing equations in fluids by these two scientists.

In these alternative co-ordinate systems the surfaces on which each co-ordinate is constant may deform with time as the fluid moves. Note too that this system is not an orthogonal co-ordinate system, as the surfaces of constant pressure are not orthogonal to those of constant x or y.

We shall need velocity components of the particles in the new system. These are defined in the obvious way:-

$$u = \frac{Dx}{Dt}, \qquad v = \frac{Dy}{Dt}, \qquad \omega = \frac{Dp}{Dt}.$$

Note that the first two components mean exactly what they mean in the height co-ordinate system; we have not resolved any vectors along the surfaces of constant coordinate as would have been the case if we were designing an orthogonal co-ordinate set. Note too that the third equation defines omega as the replacement for the vertical velocity. If the particles are moving upwards in height co-ordinates (positive w), then ω will usually be negative.

Following the same procedure as we did for $\frac{D}{Dt}$ in height co-ordinates, it is easy to show that $\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_{x,y,p} + u\left(\frac{\partial}{\partial x}\right)_{t,y,p} + v\left(\frac{\partial}{\partial y}\right)_{t,x,p} + \omega\left(\frac{\partial}{\partial p}\right)_{t,x,y}$, in which the subscripts denote which variables are held constant while the differentiation is performed. This is completely analogous to the height-co-ordinate expression. We shall drop the subscripts from now on, as it is usually obvious from the context which co-ordinate system is under consideration and write

$$\boxed{\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right) + u\left(\frac{\partial}{\partial x}\right) + v\left(\frac{\partial}{\partial y}\right) + \omega\left(\frac{\partial}{\partial p}\right)}$$

Eq 1

Of the three terms in the horizontal equation of motion, the acceleration is now easy to write down, as it is simply $\frac{D}{Dt}\mathbf{v}_h$. The Coriolis acceleration is even easier, as it is unaltered , being $f\mathbf{k} \wedge \mathbf{v}_h$. This leaves just the pressure gradient term to cast in the new co-ordinates.

The Geopotential

It will prove useful to introduce here the concept of the geopotential. This is will be denoted by φ and is defined by $d\varphi = gdz$ or, on integrating, $\varphi = \int_0^z gdz$.

NOTE that φ is one way of writing the Greek symbol *phi*. Another way of writing it is ϕ. We have used the latter form (straight phi) to mean latitude, and will try to stick to that usage, reserving φ (curly phi) for geopotential.

It is obvious from the definition that φ is the work done in raising a unit mass from mean sea level to the height z.

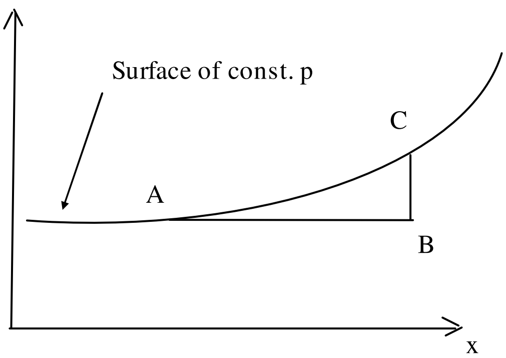
From the hydrostatic equation we clearly have that $\delta p = -\rho\delta\varphi$.

It is often convenient to express φ in terms of a height-like variable, z^* say, defined by $z^* \equiv \varphi / g_0$ where $g_0 \equiv 9.81 ms^{-2}$. z^* is called the *geopotential height*. If the geopotential is quoted in this way in terms of the geopotential height, it is often said to be expressed in *geopotential meters*.

Pressure gradient term

The x-component of the specific pressure gradient force is $-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_{t,y,z}$ in z-co-ordinates

Now the sketch shows a section parallel to the x-z axis. We see that



$$\left(\frac{\partial p}{\partial x}\right)_{t,y,z} \equiv \underset{\delta x \to 0}{Lim}\frac{\delta p}{\delta x}$$

$$= \underset{\delta x \to 0}{Lim}\frac{p_B - p_A}{\delta x} \quad = \underset{\delta x \to 0}{Lim}\frac{p_B - p_C}{\delta x}$$

(since $p_A = p_C$ as they are not on the same pressure surface).

Now by the hydrostatic equation $p_B - p_C = \rho g\delta z = \rho\delta\varphi$.

Hence the x-component of the specific pressure gradient force is $-\frac{1}{\rho}\underset{\delta x \to 0}{Lim}\frac{\rho\delta\varphi}{\delta x} = -\left(\frac{\partial \varphi}{\partial x}\right)_{t,y,p}$. Similarly, the y-component is $-\left(\frac{\partial \varphi}{\partial y}\right)_{t,x,p}$.

Thus the horizontal equation of motion can be written

$$\boxed{\frac{D}{Dt}\mathbf{v}_h + f\mathbf{k}\wedge\mathbf{v}_h = -\nabla_p\varphi}$$

or

$$\boxed{\frac{D}{Dt}\mathbf{v}_h + f\mathbf{k}\wedge\mathbf{v}_h = -g_0\nabla_p z^*}$$

Eq 2

The subscripts $_0$ and the superscript * are usually dropped with little ambiguity. We shall follow this practice.

Geostrophic wind in pressure co-ordinates

The geostrophic wind $\mathbf{v}_g$ is defined so that its horizontal Coriolis acceleration is exactly produced by the horizontal pressure gradient force. So $f\mathbf{k} \wedge \mathbf{v}_g = -g\nabla z$, giving

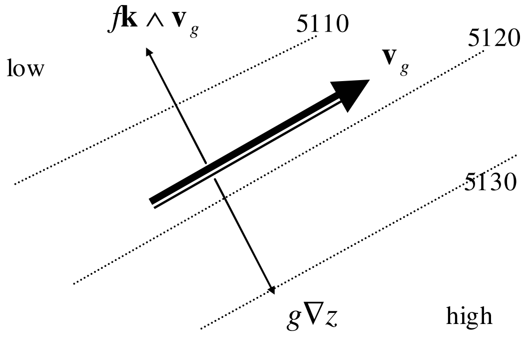
$$\boxed{\mathbf{v}_g = \mathbf{k} \wedge \frac{g}{f}\nabla z}$$

Eq 3

Just as we can draw maps showing contours, or isolinses, of pressure on a constant height surface, we can also draw maps of the height of a surface of constant pressure.

INTEND TO INSERT DIGITISED WEATHER MAP HERE. ONE IS TO BE FOUND IN THE SLIDES

The geostrophic wind is related to gradients of height of a constant pressure surface in a manner analogous to its relation to the gradients of pressure at a constant height. That is, the geostrophic wind is directed perpendicular to the gradient of height. The sense is such that in the northern hemisphere, facing in the direction towards which the geostrophic wind is pointing, low heights are on the left. Note that factor relating the magnitude of the geostrophic wind to the magnitude of the height gradient depends on latitude (through f); there is no extra variation arising from the variation of density with height as with the height co-ordinate form of the equation. For people studying weather maps without computer aids, and who are therefore using graphical tools, this is a distinct advantage, as one tool works for all pressure levels.



In this sketch, which shows a section o f a map of the 500hPa surface as it might appear in a northern hemisphere case, the dotted lines are contours (isolines of constant height), labelled in meters. The geostrophic wind is parallel to the contours and its magnitude can be calculated from the separation of the contours

Change in geostrophic wind with height

An interesting and very useful insight arises if we think about the difference in geostrophic wind on two pressure levels. Consider two pressure surfaces with pressures which we will call p_1 and p_2, with $p_1 > p_2$, so that p_2 lies above p_1. Now let their heights be z_1 and z_2. These heights are functions of time and of horizontal position, of course.

Now let $\mathbf{v}_{g1}$ and $\mathbf{v}_{g2}$ be the geostrophic wind on the respective surfaces. The vector difference between the geostrophic winds at the upper level and lower level will be denoted by $\mathbf{v}_T$ for reasons which will become apparent. Thus

$$\mathbf{v}_T = \mathbf{v}_{g2} - \mathbf{v}_{g2} = \frac{g}{f}\mathbf{k} \wedge \left(\nabla_p z_2 - \nabla_p z_1\right)$$
$$= \frac{g}{f}\mathbf{k} \wedge \nabla_p z'$$

Eq 4

We have written $(z_2 - z_1) \equiv z'$ For obvious reasons z' is called the *thickness* of the layer between pressure levels p_1 and p_2. Clearly $\mathbf{v}_T$ is related to z' in exactly the same way that $\mathbf{v}_g$ is related to z, so that the thermal wind blows parallel to the lines of constant thickness and so on.

We saw in chapter 2 that

$$z' = \frac{R\overline{T}}{g}\ln\frac{p_1}{p_2}$$

where $\overline{T}$ is the average[2] temperature of the layer. Putting this into Eq 1Eq 4 gives

$$\mathbf{v}_T = \frac{R}{f}\left(\ln\frac{p_1}{p_2}\right)\mathbf{k} \wedge \nabla\overline{T}$$

Thus $\mathbf{v}_T$ is associated with the temperature field (vertically averaged at each location) in a similar way to the way $\mathbf{v}_g$ is related to z. That is to say, $\mathbf{v}_T$ lies parallel to the isotherms (lines of equal $\overline{T}$) with cold air to the left in the Northern Hemisphere, and with a strength which is inversely proportional to the isobar spacing and to the sine of latitude. It is this relation to the thermal structure which led to the name *thermal wind* for $\mathbf{v}_T$.

$\mathbf{v}_T$ is an approximation, correct to order R_o to the difference in actual wind between two levels, just as the geostrophic wind is an approximation to the wind at a given level.

Often it is possible to see that the wind is different at different levels of that atmosphere by comparing the motion of high clouds and low clouds. From this it is possible to work out in which direction the air is warmer and which colder and hence get a qualitative forecast of whether the temperature will rise or fall in the next few hours, according to whether the warm air is blowing towards or away from the observer.

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[2] W.r.t. ln p