Chapter 5: Balance of forces in synoptic scale flow

Horizontal forces and acceleration

In this chapter we shall look first in some detail at the consequences of the scales of motion for the sizes of these forces, leading us to deduce an approximate relationship between the wind and the pressure distribution.

Our starting point is the equation of horizontal motion from the previous chapter.

$$\frac{D\mathbf{v}_h}{Dt} = -f\mathbf{k}\wedge\mathbf{v}_h - \frac{1}{\rho}\nabla_h p \qquad \textbf{Eq 1}$$

horizontal acceleration	=	specific Coriolis force	+	specific pressure gradient force

Putting these forces together (and recalling that we are only discussing horizontal components here) we get the situation shown to the right. The direction of the isobars (thin curves) and the direction of the wind (broad arrow) have been arbitrarily chosen. The specific Coriolis force is perpendicular to the wind and the specific pressure gradient force is perpendicular to the isobars. The nett resultant of these two forces produces the acceleration, shown in the sketch as the open arrow.

The Rossby number

It is useful to consider the typical magnitudes and relative sizes of the forces and the acceleration.

The specific Coriolis force has magnitude $|f\mathbf{v}_h|$, so if U is typical magnitude of the horizontal wind, the specific Coriolis force has magnitude $|f|U$. The pressure gradient force is a little trickier to estimate directly, so we will next consider the acceleration. If T is a typical timescale, then basing our estimate on $\partial\mathbf{v}_h/\partial t$ the acceleration has typical magnitude U/T. Alternatively if a typical horizontal length scale in the motion under study is L and we base our estimate on $\mathbf{v}_h\nabla\mathbf{v}_h$ we obtain the estimate U^2/L. This is consistent with the previous estimate provided that T is the time an air particle takes to traverse the system, because then $T = L/U$.

A useful measure is the Rossby number, R_o, defined as the typical ratio of the relative acceleration to the specific Coriolis force. We may expect this to be different for different scales of motion. For instance if the period of rotation of the frame of reference is very long compared with the timescale of the system, rotation will presumably be unimportant and the Rossby number would very small. Using our previous estimates we obtain R_o = (typical accel)/(typical specific Coriolis force) = $\left(U^2/L\right) \div \left(Uf\right)$, giving

$$\boxed{R_0 = \frac{U}{Lf}}.$$

That is the general expression. What is its magnitude for a typical mid-latitude weather system? These systems are often said to be "of synoptic scale". This name was coined when weather maps were analysed by hand. The practice grew of plotting all the observations for *a given time* on a regional chart (say Europe and the North Atlantic). This was known as the "synoptic chart" (Greek for "simultaneous view"). These charts are dominated in middle and high latitudes by the familiar cyclones and anticyclones (areas of low and high pressure) which control our weather. For such systems we observe that typically $U \approx 10ms^{-1}$ and $L \approx 1000km = 10^6 m$. We noted in the last chapter that $f \approx 10^{-4} s^{-1}$, so that

$$R_0 \approx \frac{1}{10}.$$

That is, the relative acceleration is an order of magnitude smaller than the specific Coriolis force. This can only happen if the Coriolis force and pressure gradient force are approximately in balance (i.e. equal and opposite), so that their nett resultant is small.

Note that we have not yet given an explanation of why the Coriolis force and pressure gradient force are approximately equal and opposite, we have merely noted that they must be to be consistent with the typical space and timescales of the phenomena.

The Geostrophic Wind

The approximate balance between the Coriolis force and the pressure gradient force in middle latitudes is so important that it has proved useful to introduce a reference or hypothetical wind, $\mathbf{v}_g$,called the *geostrophic* wind, for which the balance is exact. Thus the geostrophic wind is defined, for a given pressure gradient, as that wind whose Coriolis force is equal and opposite to the pressure gradient force. [Geostrophic come from the Greek for "earth turning"].

Thus $-f\mathbf{k} \wedge \mathbf{v}_g - \frac{1}{\rho}\nabla_h p \equiv 0$, or on re-arranging

$$\boxed{\mathbf{v}_g \equiv \frac{1}{\rho f}\mathbf{k} \wedge \nabla_h p}$$

It is clear that the geostrophic wind is parallel to the isobars and that if you stood with your back to the geostrophic wind in the northern hemisphere, low pressure would be on your left, while in the southern hemisphere it would be on your right (as f has the opposite sign there). The strength of the geostrophic wind is inversely proportional to the isobar spacing. For a given pressure gradient (or isobar spacing) the geostrophic wind increases towards low latitudes as it is inversely proportional to f.

The usefulness of the geostrophic wind is that it is a reasonable approximation to the actual wind (obviously correct to order of the Rossby number). Thus a chart of isobars on a surface of constant height allows us to visualise and to calculate the true wind to a reasonable approximation.

Note that we have neglected frictional forces, so, in the lowest few hundred meters of the atmosphere, where friction with the ground may be appreciable, the geostrophic approximation (that $\mathbf{v} \approx \mathbf{v}_g$) is not so good as it is aloft.

The geostrophic approximation becomes poorer as the equator is approached (for systems of the given space and velocity scales), because f becomes smaller and R_o larger. At and near to the equator, where f is zero, the Coriolis force is negligible and the relative acceleration must equal the specific pressure gradient force.

The ageostrophic part of the wind

The difference between the true wind and the geostrophic wind is known as the *ageostrophic wind component* or simply as the *ageostrophic wind*. If we denote this by v_a, then $\mathbf{v}_a \equiv \mathbf{v} - \mathbf{v}_g$ and

$$\frac{D\mathbf{v}_h}{Dt} + f\mathbf{k} \wedge \mathbf{v}_g + f\mathbf{k} \wedge \mathbf{v}_a = -\frac{1}{\rho}\nabla_h p \,.$$

Hence,

$$\boxed{\mathbf{v}_a = \frac{1}{f}\mathbf{k} \wedge \left(\frac{D\mathbf{v}_h}{Dt}\right)}.$$

Thus the ageostrophic wind is perpendicular to the acceleration and directed to the left from it in the Northern Hemisphere (right in the Southern Hemisphere).

Estimating the acceleration from the pressure field

For synoptic scale motion in middle latitudes, the geostrophic approximation can be used in the expression for the acceleration, thus:-

$$\frac{D\mathbf{v}_h}{Dt} = \frac{\partial \mathbf{v}_h}{\partial t} + \mathbf{v}_h.\nabla\mathbf{v}_h + w\frac{\partial \mathbf{v}_h}{\partial z} = \frac{\partial \mathbf{v}_g}{\partial t} + \mathbf{v}_g.\nabla\mathbf{v}_g + w\frac{\partial \mathbf{v}_g}{\partial z} + O(R_o)$$
$$= \frac{\partial \mathbf{v}_g}{\partial t} + \mathbf{v}_g.\nabla\mathbf{v}_g + O(R_o)$$

The final step in the above reasoning (i.e. dropping the term in the vertical velocity) is not self-evident and will be justified in a later chapter. That step too is correct to order of the Rossby number.

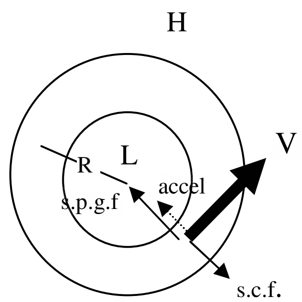
Since we can find the acceleration in terms of the pressure field, we can use this information about the acceleration to give a more accurate estimate of the true wind than is afforded by the simple use of the geostrophic approximation.

Steady circular flow

There are some situations in which we can write down an exact expression for the acceleration in terms of the real wind and solve for the windspeed. A special case, which has some resemblance to commonly occurring flow patterns, is that of steady circular flow round circular isobars. This has come to be known in the literature as the *gradient wind* as it was the earliest more accurate (than geostrophic) estimates of the windspeed for a given pressure gradient.

Cyclonic case

We consider first the cyclonic case. The sketch represents concentric isobars with low pressure in the centre. In the Northern hemisphere The flow can be expected to be clockwise round the low. The specific pressure gradient force is directed towards the centre of the low, and the specific Coriolis force is directed perpendicular to the wind and is outwards.



The acceleration is produced by the vector sum of the two forces, which in this case is directed inwards and has magnitude V^2/R , where V is the windspeed and R is the distance to the centre of the cyclone.

The specific pressure gradient force has magnitude $\frac{1}{\rho}|\nabla p|$, but it is convenient to write this as fV_g, where V_g is the magnitude of the geostrophic wind. The specific Coriolis force has magnitude fV.

Thus we have

$$\frac{V^2}{R} = fV_g - fV$$

Eq 2

Since the left hand side of this equation is always positive, we shall have that the geostrophic wind is greater than the actual wind in this case. i.e. *the geostrophic approximation over-estimates the wind in the cyclonic case*[1].

Eq 2 is a quadratic in the windspeed which has solutions

$$V = \frac{-Rf \pm \sqrt{R^2 f^2 + 4RfV_g}}{2}.$$

We appear to have found two solutions. Clearly the one we want should tend towards the geostrophic wind as R tends to infinity, since this leads to straight isobars and negligible acceleration. Moreover this will obviously be the solution from the + sign, as the other solution is entirely negative, which would be in the wrong direction. Thus the gradient wind in the cyclonic case (low pressure centre case) is given by

$$\boxed{V = \frac{-Rf + \sqrt{R^2 f^2 + 4RfV_g}}{2}}$$

The solution with the – sign that we rejected would, as already stated, be flowing clockwise (anti-cyclonically) round this low pressure, which is never observed in practice. It can be shown, by an analysis too advanced for this course that, this anomalous flow would be unstable. That is, although the forces balance, a slight perturbation would lead to an exponentially growing lack of balance in a way which destroys the circular flow. In the true physical solution with the + sign, any small perturbation produces changes in the forces which tends to restore the balance.

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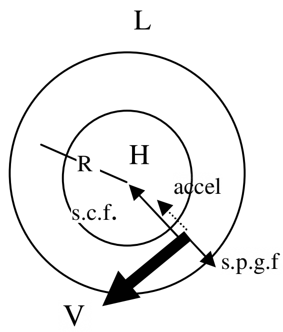
[1] (Note that if this remains true in the Southern hemisphere despite the change in sign of f, because we would have to draw the diagram differently in the southern hemispheric case. An alternative way of thinking about it is to keep the same diagram but then the windspeeds would be negative quantities because they would be directed in the opposite sense to what we have drawn here)

Anticyclonic case

We turn now to the anticyclonic case. In principle we can obtain the details of this case by simply changing the sign of V_g in the foregoing and then looking at the solution of the quadratic bearing in mind that we expect the flow to be clockwise. However it is probably easier to redraw the diagram and hence redefine the sign conventions as in the figure to the right. Now we have



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$$\frac{V^2}{R} = fV - fV_g$$

with solutions

$$V = \frac{Rf \pm \sqrt{R^2 f^2 - 4RfV_g}}{2}.$$

To choose the correct sign we need $V \to V_g$ as $R \to \infty$. It is left as an exercise to show that this means that the – sign is needed, so that the solution is

$$\boxed{V = \frac{Rf - \sqrt{R^2 f^2 - 4RfV_g}}{2}}$$

Eq 3

Again the anomalous solution with the other sign can be shown to be unstable.

Considering Eq 3, it is apparent that to obtain real roots, we need

$$Rf > 4V_g \qquad \text{or} \qquad V_g < \frac{Rf}{4}.$$

Thus in the anticyclonic case there is a limit to the pressure gradient.

At the centre of anticyclones, pressure gradients are small, while in cyclones there is no such limit. There are no limits either in tornadoes and hurricanes. In these the Coriolis force is, of course, negligible and the centripetal acceleration is produced entirely by the pressure gradient force.