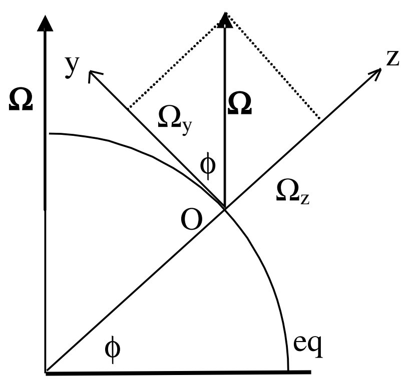
Chapter 4: The Tangent Plane approximation

The Tangent Plane Approximation

If we regard the Earth as a sphere, a natural co-ordinate system for writing down the component form of the equations of motion would be spherical co-ordinates (say longitude, latitude and radial distance)[1]. For weather forecasting and climate prediction this would be essential. However, that leads to levels of complexity of the mathematics which are too great for this course, in which we wish mostly to get some insight into the physical principles at work in weather systems and these can be illustrated in an approximate system which can be treated more simply.

The approximation we shall use is known as the *tangent plane approximation*, as the idea is to treat the earth as locally flat. That is, we consider that the earth's surface in the vicinity of a point stays close to the tangent plane. Take a position O on the surface of the Earth. Then we shall take rectangular co-ordinates Oxyz such that Oxy is the plane which is tangential to the Earth's surface at O, with Ox directed eastward, Oy to the North and Oz directed vertically upwards at O. That can, of course, be done with no approximation. In the tangent plane approximation we assume that the surface of the Earth in the vicinity of O remains in the Oxy plane and that apparent gravity is directed parallel to Oz (but downwards) even at finite x, y displacements from the origin. We further assume that on the lines parallel to Ox and Oy are lines of latitude and longitude respectively, so that latitude and longitude form a regular recangular grid on the tangent plane. (Contrast that with the situation on a stereographic projection where the lines of latitude and longitude are determined by



[1]Going to greater levels of sophistication we could recognise that the earth is an oblate spheroid and use a co-ordinate frame suitable for those (a series of spheroids and hyperboloids of revolution), though in practice that is not worth the effort as other approximations are needed to the physics which make this extra accuracy pointless.

projection from the antipodal point onto the tangent plane. In that projection the lines of latitude are circles and the lines of longitude converge.)

We shall write the unit vectors in the direction of the x,y,z directions as $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and the components of as (u, v, w). Then we have that

$$\frac{D}{Dt}\mathbf{v} = \mathbf{i}\frac{D}{Dt}u + \mathbf{j}\frac{D}{Dt}v + \mathbf{k}\frac{D}{Dt}w$$

and

$$\wedge \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Omega_x & \Omega_y & \Omega_z \\ u & v & w \end{vmatrix}$$

If ϕ is latitude, then it is clear from the sketch and simple geometry that

$$= (\Omega_x, \Omega_y, \Omega_z) = \Omega(0, \cos\phi, \sin\phi)$$

So that

$$\begin{aligned} \wedge \mathbf{v} = {} & \mathbf{i}(w\Omega\cos\phi - v\Omega\sin\phi) \\ & + \mathbf{j}(u\Omega\sin\phi - w0) \\ & + \mathbf{k}(0.v - u\Omega\cos\phi) \end{aligned}$$

$\frac{1}{\rho}\nabla p$ has components $\frac{1}{\rho}\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right)$ and $\mathbf{g}$ is vertically downwards (that is how we define the vertical of course). So $\mathbf{g} = (0,0,-g)$.

Putting all these together gives

x-momentum equation

$$\frac{Du}{Dt} + 2\Omega w\cos\phi - 2\Omega v\sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

y-momentum equation

$$\frac{Dv}{Dt} + 0 + 2\Omega u\sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

z-momentum equation

$$\frac{Dw}{Dt} + 0 - 2\Omega u\cos\phi = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

Approximations for large-scale flow

When these equations are applied to synoptic-scale flow in mid-latitudes, we find that some of the terms can be neglected. The timescale of such flows is around a day (i.e. in the range of a significant fraction of a day to several days), while the horizontal velocities are around $10ms^{-1}$ and the distances involved are of order $1000km$. The timescale just quoted is based on the observed lifetimes of systems, but it is also typical of the time taken for an air parcel to flow through the system, viz spacescale/velocity $= 1000km/10ms^{-1} = 10^5 s \sim 1day$. [Exercise, how close is one day to $10^5 s$?]

Weather systems occupy the troposphere which is about 10km deep, so on this argument we might expect vertical velocities to be typically $10km \div (10^5 s) = 10cms^{-1}$. It turns out, for reasons which will be discussed in a later chapter, that large-scale vertical velocities are about an order of magnitude less than this, i.e. about one km per day or $1cms^{-1}$.

Hence we note that in the x-momentum equation the term $2\Omega w \cos\phi$ is smaller than $2\Omega v \sin\phi$ by the ratio $\frac{w}{v} \sim \frac{10^{-2}ms^{-1}}{10ms^{-1}} \sim 10^{-3}$ and can be ignored to high accuracy, allowing us to write the two horizontal components of the momentum equations as

$$\frac{Du}{Dt} - fv = -\frac{\partial p}{\partial x}$$
$$\frac{Dv}{Dt} + fu = -\frac{\partial p}{\partial y}$$

Eq 0 (a & b)

Where $f \equiv 2\Omega \sin\phi$ is called the *Coriolis parameter*. It is twice the vertical component of the earth's rate of rotation.

Sometimes it is useful to write this pair of equations as a single equation for the horizontal velocity $v_k \equiv (u, v, 0)$ and the horizontal gradient operator $\nabla_h \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right)$, giving

$$\frac{D\mathbf{v}_h}{Dt} \quad + \quad f\mathbf{k} \wedge \mathbf{v}_h \quad = \quad -\frac{1}{\rho}\nabla_h p$$

Eq 0

horizontal + Coriolis = pressure gradient acceleration   Accel   force

Turning now to the vertical component we see that $\frac{Dw}{Dt} \sim \frac{10^{-2} ms^{-1}}{10^{5} s} = 10^{-7} ms^{-2}$, which is about 8 orders of magnitude less than g. Now in middle latitudes $2\Omega\cos\phi$ has the same order of magnitude as f; indeed they are the same at lat 45 where they are closely equal to $10^{-4} s^{-1}$, so that $2\Omega u\cos\phi$ has magnitude $10^{-3} ms^{-2}$. This is 4 orders of magnitude less than g. Hence if we neglect these two small terms in comparison with g we are left with

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

or

$$\frac{\partial p}{\partial z} = -\rho g$$

Eq 0

which we recognise as the hydrostatic equation. Thus in synoptic scale flow, the atmosphere is in hydrostatic balance to about one part in ten thousand.

Note that we have not said that there are no vertical accelerations, we have said that it is small. So small in fact that we can calculate the pressure distribution from the density distribution to high accuracy. If we calculate the fluid motion using a set of equations which have the vertical momentum equation replaced by Eq 0, then we clearly will not be able to use that equation to calculate the vertical acceleration. Provided a consistent set of approximations have been made, it will, however, still be possible to deduce the vertical acceleration indirectly.

Horizontal forces and acceleration

We look now in more detail at the way the forces combine to produce the acceleration. Then we shall contemplate the consequences of the scales of motion for the sizes of these forces, leading us to deduce an approximate relationship between the wind and the pressure distribution.

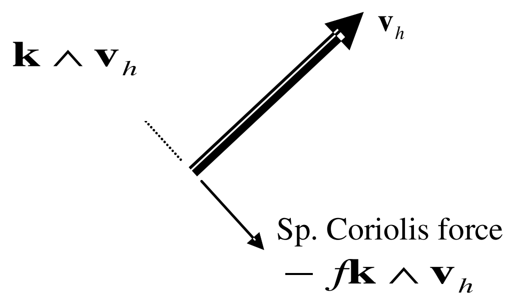
Our starting point is the equation of horizontal motion from the previous chapter. We recast this slightly by transferring the Coriolis term to the right hand side of the equation, leaving just the relative acceleration on the left hand side. If we now regard the terms on the right as forces which produce this relative acceleration this Coriolis term is now thought of as a force. This change of description is analogous to the way we talk sometimes of a centrifugal *force* but at other times of a centripetal *acceleration*, say in cornering vehicles, according to which point of view is convenient.

$$\frac{D\mathbf{v}_h}{Dt} = -f\mathbf{k}\wedge\mathbf{v}_h - \frac{1}{\rho}\nabla_h p$$

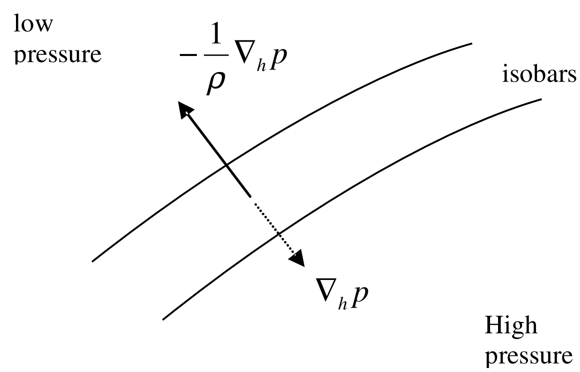
Eq 0

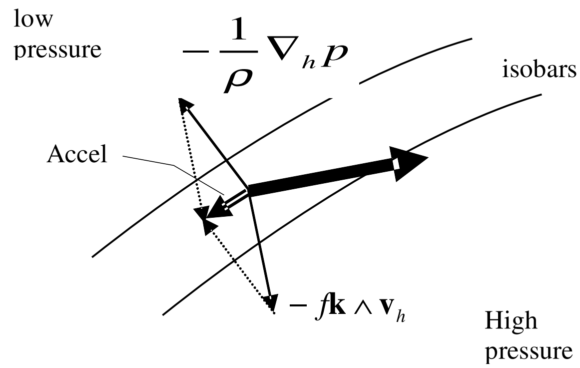
horizontal acceleration	=	specific Coriolis force	+	specific pressure gradient force

In the sketch the thick arrow shows the horizontal wind vector. Using the right hand corkscrew rule, we see that $\mathbf{k}\wedge\mathbf{v}_h$ is directed at right angles to the left of the wind. Noting that f is positive in the northern hemisphere, we see that the specific Coriolis force is therefore directed perpendicular to the wind and to the right in the northern hemisphere. As f is of the opposite sign in the southern hemisphere, the sense of the specific Coriolis force is directed to the left of the wind in that hemisphere.



The sketch to the right shows the configuration of the pressure field and the specific pressure gradient force. This is directed perpendicular to the isobars and is directed towards low pressure.





Putting these forces together (and recalling that we are only discussing horizontal components here) we get the situation shown to the right. The direction of the isobars (thin curves) and the direction of the wind (broad arrow) have been arbitrarily chosen. The specific Coriolis force is perpendicular to the wind and the specific pressure gradient force is perpendicular to the isobars. The nett resultant of these two forces produces the acceleration, shown in the sketch as the open arrow.