Chapter 3: The Equations of Motion

Content of the chapter

The aim of this chapter and the next is to deduce the appropriate equations of motion for large-scale flow in the atmosphere. This means writing Newton's Laws of Motion (force=mass x acceleration) in a form where it can be applied in a fluid medium. In addition we shall need to do the same for the related laws which express that matter is conserved, and the constraints of thermodynamics.

Clearly we shall be writing rates of changes of things. This brings a subtlety to consider, because we shall sometimes need the rate of change measured at a fixed point, and sometimes we shall require the rate of change experienced by the fluid particles as they move about.

The material derivative

The rate of change experienced by fluid particles is given the special symbol $\frac{D}{Dt}$ and is known as the *material derivative*.

Any quantity s will be a function of three position co-ordinates, x, y, z and of time t, i.e. $s = s(t, x, y, z)$. If s changes by δs as the independent variables change by $(\delta t, \delta x, \delta y, \delta z)$, then

$$\delta s = \frac{\partial s}{\partial t}\delta t + \frac{\partial s}{\partial x}\delta x + \frac{\partial s}{\partial y}\delta y + \frac{\partial s}{\partial z}\delta z$$

To calculate the change experienced by a fluid particle, we need to choose $(\delta x, \delta y, \delta z)$ to be consistent with the displacement of the particle in the time interval. If we denote the velocity of the fluid by $\mathbf{v}$ with components given by (u, v, w), then

$$(\delta x, \delta y, \delta z) = (u, v, w)\delta t$$

which means that

$$(\delta s)_{particle} = \left(\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} + w\frac{\partial s}{\partial z}\right)\delta t$$

By the definition of what we mean by $(\delta s)_{particle}$, We have

$$\frac{Ds}{Dt} = \lim_{\delta t \to 0}\frac{(\delta s)_{particle}}{\delta t}$$

Dividing by δt and taking the limit of small time interval leads to

$$\frac{Ds}{Dt} = \left(\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} + w\frac{\partial s}{\partial z} \right)$$

or

$$\boxed{\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}.\nabla}$$

Advection

Some properties of the individual air particles will be unchanged as they move. For instance if we introduce an inert gas into an air sample, then the mixing ratio of that gas in a given air parcel will to all intents and purposes remain unchanged as the air parcel moves about. Of course there are limits to this idea; molecular diffusion will eventually allow the gas to "leak" out into neighbouring particles, and if we try to regard too big a lump of air as our parcel, then motion systems on a smaller scale may tear it apart. But let us consider the ideal case where these smaller scale motions and molecular diffusion are negligible. If we denote the mixing ratio of the gas by s, then the statement that the mixing ratio of the inert gas does not change for individual air parcels is clearly that

$$\frac{Ds}{Dt} = 0$$

It follows that

$$\frac{\partial s}{\partial t} = -\mathbf{v}.\nabla s$$

The left hand side here is the local rate of change of mixing ratio. That is, it is the rate of change which would be measured for instance by a measuring instrument fixed at a geographical location. The right hand side involves the wind and spatial gradients of the mixing ratio. If s increases in the direction of the wind, then the right hand side will be negative. So if the mixing ratio is unchanged for individual air parcels, the local rate of changed measured at a fixed point will decrease in this case, as air of lower mixing ratio is blown over the station. In general the local rate of change for quantities which do not change on that air particle is given by $-\mathbf{v}.\nabla s$. This quantity is known as *the advection*, from Greek words meaning "carried to".

Continuity of Mass

We shall need a statement that matter is neither created nor destroyed. Consider a small lump or parcel of fluid with volume $\Delta\tau$ and density ρ. Then the equation of continuity of mass (i.e. its non-destruction) expresses the statement that following the fluid particles the product of those two quantities is constant.

$$\frac{D}{Dt}(\rho\Delta\tau)=0$$

Eq 1

This form of the equation will suffice for now, but it will be further developed in a later chapter.

Newton's first law

We now begin to develop an expression of Newton's first law of motion, rate of change of momentum -= forces. Consider a "lump" of fluid with volume τ composed of small elements $\Delta\tau$. The rate of change of momentum of an elemental "chunk" is

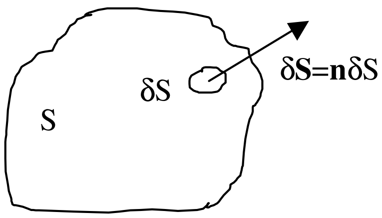
$$\frac{D}{Dt}(mass\times velocity)=\frac{D}{Dt}(\rho\Delta\tau.\mathbf{v})=\rho\Delta\tau\frac{D\mathbf{v}}{Dt}+\mathbf{v}\frac{D}{Dt}(\rho\Delta\tau)=\rho\Delta\tau\frac{D\mathbf{v}}{Dt}+0 \text{ by Eq 1}$$

So integrating over all elements, to give the rate of change of momentum for the whole lump

$$\text{Rate of change of momentum}=\int_\tau \rho\frac{D\mathbf{v}}{Dt}d\tau$$

If we neglect frictional (viscose) forces for now, there are two forces acting on the lump, namely pressure and gravity.

To calculate the pressure force, let $\delta\mathbf{S}$ be a vector associated with an element of surface area dS directed along the outward normal $\mathbf{n}$, so that $\delta\mathbf{S}=\mathbf{n}.\delta S$, where $\mathbf{n}$ is a unit vector perpendicular to the surface of the lump directed outwards.



The force due to pressure on the element dS due to the fluid outside the lump pressing inwards onto the lump is $-p\delta\mathbf{S}$, the minus sign arising because the inward force is in the opposite direction to $\mathbf{n}$.

Thus the force on the whole lump due to the pressure forces is $-\oint_s p d\mathbf{S}$, where the surface integral is over the whole surface area.

Now Gauss's theorem states that for an arbitrary quantity A, $\oint A d\mathbf{S}=\int_\tau \nabla A d\tau$. Hence the pressure force can be written $-\int_\tau \nabla p d\tau$.

The force due to gravity on an element is $\rho\delta\tau.\mathbf{g}_t$ where $\mathbf{g}_t$ is the gravitational acceleration. It has been given a subscript t to distinguish it from the apparent gravity which we shall introduce below. Thus the total gravitational force on the lump is $\int_\tau \rho \mathbf{g}_t d\tau$.

Equating the rate of change of momentum to the forces and re-arranging slightly gives

$$\int_\tau \left\{ \rho \frac{D\mathbf{v}}{Dt} + \nabla p - \rho \mathbf{g}_t \right\} d\tau = 0$$

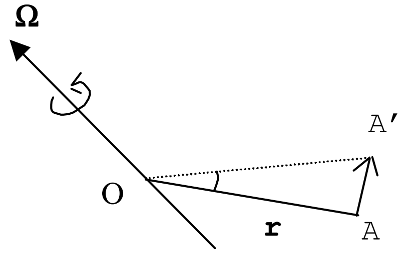
Now this equation is true for all lumps of fluid. This can only happen if the quantity in the curly brackets is zero. i.e. (on re-arranging slightly),

$$\boxed{\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g}_t}$$

This is our version of one of the Navier-Stokes equations of fluid dynamics. (Others are continuity equation etc. The full Navier-Stokes equations contain viscous effects which we have neglected.) We have implicitly written this equation in an inertial frame of reference. In meteorology it is more convenient to write it in a frame of reference rotating with the earth. Applying the appropriate transformation is the subject of the next sections.

The frame of reference which we used in the last chapter to write down the equations of motion was assumed to be inertial (non-accelerating) in the sense of classical physics. However the most convenient frame for us to use is one fixed relative to Earth. As the systems we shall be studying last from 1 to several days, during which time Earth will rotate a few times relative to an inertial frame, the Earth-frame is not inertial and we shall need to recast the momentum equation to take into account that rate of change of vector quantities are different in frames of reference which are accelerating relative to each other.

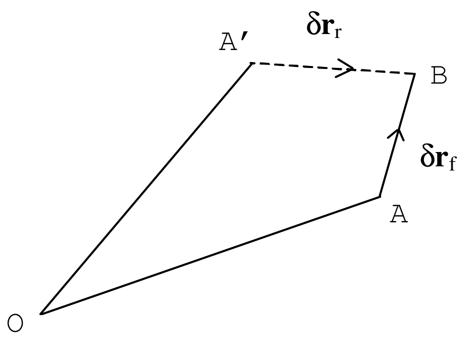
As the first stage of deriving the appropriate equations, consider how a position in one frame seems to change in another frame rotating with respect to the first one.



In the sketch $\mathbf{\Omega}$ is the rate of rotation vector of the rotating frame relative to the fixed frame. It is defined to be pointing along the axis of rotation in the sense of a right-handed corkscrew. The rate of angular rotation is $|\mathbf{\Omega}|$.

Now consider two points, one, labelled **A**, fixed in the fixed frame, and the other, **A′**, fixed in the rotating frame. Let them be co-incident at time t. The sketch shows the situation as it appears from the fixed frame at time t+δt. A is fixed and in the same position, but in the other frame **A** appears to have moved to **A′**. The displacement **AA′** is $\mathbf{\Omega} \wedge \mathbf{r}\delta t$, where $\mathbf{r}$ is the vector from the origin to

A. It follows that seen from the rotating frame **A** appears to have moved backwards by the same magnitude. i.e. a point at $\mathbf{r}$, fixed in the fixed frame, moves $-\mathbf{\Omega}\wedge\mathbf{r}\delta t$ as seen from the rotating frame.



Now consider a particle which is at position **A** at time t but which moves to the new postion **B** by $t+\delta t$. The change in position seen from the fixed frame frame we will denote by $\delta\mathbf{r}_f$ and seen from the rotating frame we will denote by $\delta\mathbf{r}_r$.

Now

$$\delta\mathbf{r}_f = \overrightarrow{AB} = \overrightarrow{AA'} + \overrightarrow{A'B} = \delta\mathbf{r}_r + \mathbf{\Omega}\wedge\mathbf{r}\delta t$$

Dividing by δt and taking the limit gives

$$\left(\frac{D\mathbf{r}}{Dt}\right)_f = \left(\frac{D\mathbf{r}}{Dt}\right)_r + \mathbf{\Omega}\wedge\mathbf{r}$$

Although we conducted the previous argument in terms of the position vector, it is easily seen that the rate of change of any vector leads to a similar expression, so that expressed in the form of operators we have

$$\boxed{\left(\frac{D}{Dt}\right)_f = \left(\frac{D}{Dt}\right)_r + \mathbf{\Omega}\wedge}$$

Now the velocity vector $\mathbf{v}$ is simply $\dfrac{D\mathbf{r}}{Dt}$, so that the left hand side of the Navier-Stokes equation is

$$\begin{aligned}\left(\frac{D}{Dt}\right)_f\left(\frac{D}{Dt}\right)_f\mathbf{r} &= \left\{\left(\frac{D}{Dt}\right)_r + \mathbf{\Omega}\wedge\right\}\left\{\left(\frac{D}{Dt}\right)_r + \mathbf{\Omega}\wedge\right\}\mathbf{r}\\ &= \left(\frac{D}{Dt}\right)_r^2\mathbf{r} + 2\mathbf{\Omega}\wedge\left(\frac{D}{Dt}\right)_r\mathbf{r} + \mathbf{\Omega}\wedge(\mathbf{\Omega}\wedge\mathbf{r})\\ &= \left(\frac{D}{Dt}\right)_r\mathbf{v}_r + 2\mathbf{\Omega}\wedge\mathbf{v}_r + \mathbf{\Omega}\wedge(\mathbf{\Omega}\wedge\mathbf{r})\end{aligned}$$

The last term is a function of position alone, and can be easily seen to be the centripetal acceleration.

Thus, after a little more manipulation, the Navier-Stokes equation can be written

$$\left(\frac{D}{Dt}\right)_r\mathbf{v}_r + 2\mathbf{\Omega}\wedge\mathbf{v}_r = -\frac{1}{\rho}\nabla p + \{\mathbf{g}_t - \mathbf{\Omega}\wedge(\mathbf{\Omega}\wedge\mathbf{r})\}.$$

The term in the curly brackets is usually called *apparent gravity*. It is a mixture of the true gravity and the centrifugal force. In future we shall drop the subscript r for the rotating frame, and simply assume it. In addition we shall write $\mathbf{g}$ for the apparent gravity. Thus the momentum equation becomes

$$\boxed{\left(\frac{D}{Dt}\right)\mathbf{v} + 2\mathbf{\Omega} \wedge \mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{g}}$$