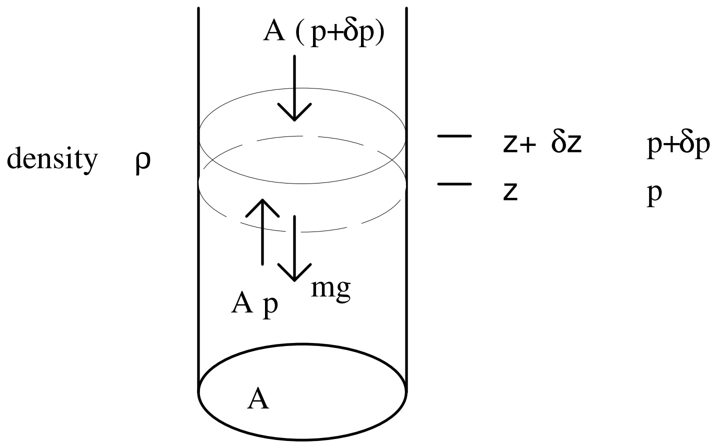
Section 2 Vertical Distributions

Variation of pressure with height

To a good approximation the atmosphere is in hydrostatic equilibrium i.e. the vertical accelerations are very small compared with the acceleration due to gravity. This allows us to relate the change in pressure with height to the local temperature as follows.

Consider a column of atmosphere as in figure extending upwards from a horizontal area A.



Let the pressure at height z be p and that at height $z+\delta z$ be $p+\delta p$. Consider the elemental slab of atmosphere between these two heights. This is being pulled downwards by a gravitational force which has magnitude (mass) x g. Where g is the gravitational acceleration. Now mass=vol x density $= A.\delta z.\rho$ so the downward force is $A.\delta z.\rho.g$. This is balanced by the pressure forces. The bottom of the slab is subjected an upward force (pressure x area) of $p.A$ exerted by the fluid below, while the top of the slab is subjected to a downward force of $(p+\delta p).A$ exerted by the fluid above. Thus the nett upward force due to these pressures is $-\delta p.A$. Equating the nett upward pressure force to the downward gravitational force and cancelling the area gives $\delta p = -\rho g \delta z$. On taking limits we obtain the *hydrostatic equation*:-

$$\frac{dp}{dz} = -\rho g$$

Eq 1

Sometimes you may see the left hand side of this equation written as a partial rather than total derivative, as a reminder that pressure varies in the horizontal as well as the vertical. Indeed this will be necessary later in this course.

A consequence of hydrostatic balance is that the pressure force at any height is the integrated weight of the atmosphere above that height.

On substituting for ρ from the gas law (**Error! Reference source not found.**) we obtain after some simple manipulation:-

$$\frac{d \ln p}{dz} = -\frac{g}{RT}$$

Integrating from height z_1 where the pressure is p_1 to z_2 where it is p_2 gives

$$p_2 = p_1 \exp - \int_{z_1}^{z_2} \frac{dz}{H}$$

where $H = \dfrac{RT}{g}$.

In general H is a function of height, because T is. H is called the *pressure scale height*. In the case of an isothermal atmosphere, in which T and hence H is constant with respect to height, and assuming for now that g does not vary with height over the ranges we are interested in, it is seen that the pressure decreases exponentially with height, falling by a factor of e as height increases by H. For a median temperature of 250K, H is 7.32 km. In such an atmosphere the pressure falls by a factor of 10 as height increases by 16.8km. By the gas law, density is proportional to pressure in an isothermal atmosphere, so it too varies exponentially.

The atmosphere is not isothermal, so the variation of pressure with height is not quite so simple, but the temperature varies by only a few tens of percent from the average, so the pressure still falls approximately exponentially with height, as does the density. As a fairly good rule of thumb the pressure fall by a factor of 10 for each 16 km of height increase. This means that 90% of the atmospheric mass lies below 16km, 99% below 32km and so on.

In these equations consistent units must be used. The S.I. unit for pressure is the Pascal (denoted Pa) which is 1 Nm^{-2}. Meteorologists have traditionally used the *bar* or more usually the *millibar* (mbar) defined as 1mbar=100Pa=1hPa. There is a trend to quoting pressures in hPa in meteorological literature.

At mean sea level the pressure is typically 1000 hPa. It may vary between about 940 and 1040 hPa.

Importance of compressibility

In the previous lecture we saw that in certain parts of the atmosphere the temperature decreased with height. In some fluids with which we are familiar that would be a strange state of affairs. In water for instance it is easy to get warm water to lie over cold, but not the other way round, as may be readily tested while in the bath. The reason is that for an incompressible fluid like water, density depends only on temperature, warm water being less dense. Thus warm water is buoyant with respect to cold water and will tend to accelerate up through it. For a compressible fluid (i.e. a gas), in contrast, the relationship between density and temperature depends on the pressure as well, and this means that air parcels moving in the vertical will change their density because of pressure changes. We explore these matters in this chapter. We shall begin with some thermodynamic considerations, because it makes a difference whether heat is supplied to the air parcels or not.

Adiabatic Changes

Any process in which no external energy is transferred to the air parcel is said to be *adiabatic* while one in which external energy is supplied is said to be *diabatic*. In the atmosphere heat can be transferred into an air parcel by radiative process and by conduction. However radiation typically produces temperature changes of about 1.5°C per day. Many temperature changes occur much faster than that, so that to a good first approximation the radiative effects can be neglected for these. Likewise molecular conduction is negligible except very near to the surface (i.e. within a millimetre or so). Consequently a whole range of processes in the atmosphere are effectively adiabatic.

Most textbooks on thermodynamics demonstrate that entropy, S, can be written

$$S = C_p \ln\left(Tp^{-\kappa}\right) + C$$

where C is a constant and $\kappa \equiv R / C_p$. Now $C_p = C_v + R$, so $\kappa = (\gamma - 1)/\gamma$ where γ is the ratio C_p / C_v. For diatomic gases γ=1.4, giving κ=2/7.

Since the entropy of an air parcel S is constant as that particle undergoes an adiabatic change, so is $Tp^{-\kappa}$. It is usual to write this constant value in the form $\theta p_0{}^{-\kappa}$, where p_0 is a reference pressure always taken to be 1000hPa. Thus

$$\theta = T\left(\frac{p_0}{p}\right)^{\kappa} \quad \text{and} \quad \mathrm{T} = \theta\left(\frac{\mathrm{p}}{\mathrm{p}_0}\right)^{\kappa}$$

θ is called the *potential temperature*. It can be thought of as the temperature an air parcel would attain if it were brought to a pressure of 1000 hPa adiabatically. For any pair of temperature and pressure values there is a corresponding value of potential temperature. During adiabatic motion the pressure and temperature of an air parcel change in such a way as to conserve potential temperature.

It will be useful later to note that for two air samples with the same pressure, their temperatures will be proportional to their potential temperatures, and their densities will be inversely proportional to their potential temperatures.

Static stability

We are now in a position to consider the criterion for static stability in an compressible fluid like air. In this section we confine our attention to processes in which water vapour does not condense out into water drops. Consider the atmospheres shown in Figure 1

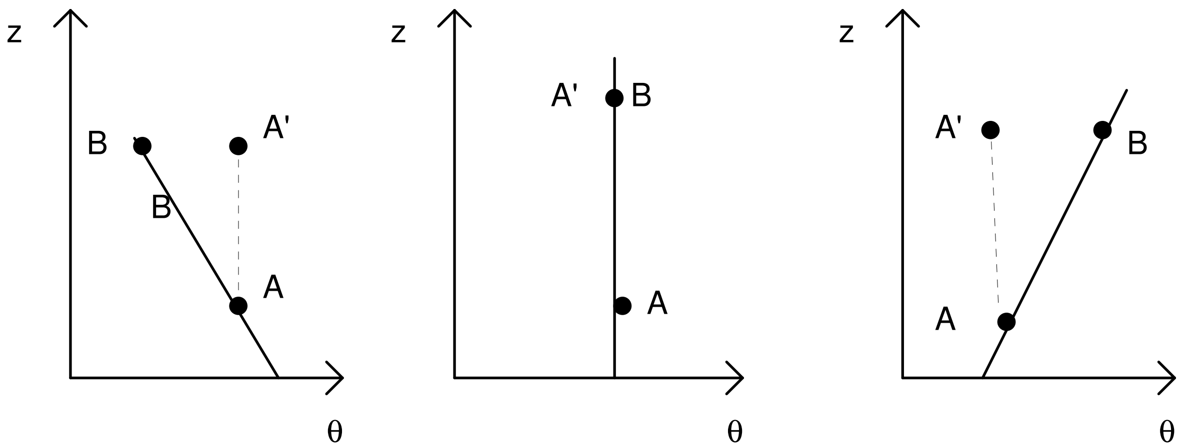


Figure 1: Unstable, neutral and stable atmospheres

Concentrate first on the left-hand distribution and imagine some process which can take an air parcel from A and raise it to some other height, shown as A' in the figure, with its pressure always adjusting to be the same as that of the surroundings. (This kind of lifting might be produced by the air having to climb a mountain for instance, or it might be produced by the mechanical stirring processes produced by the general turbulence in the boundary layer near the surface (i.e. in the lowest few hundred meters). When the particle arrives at A' it will have the same value of θ as it did at A. However the surroundings have a potential temperature as shown at B and this is lower than the value at A'. The parcel is thus less dense than its new surroundings and will be subject to a buoyancy force which will make it accelerate upwards. Thus once the particle begins to rise it continues to do so with increasing speed.

If a similar lifting process is brought about in the middle distribution, when the parcel which starts at A reaches A', it has exactly the same potential temperature as its new surroundings and hence has the same density as them. Consequently it has a no tendency to accelerate or decelerate.

In the right-hand case, when the parcel arrives at A', it is surrounded by air indicated by B in the figure, and this is potentially warmer. Thus the parcel is more dense than its new surroundings and the buoyancy forces will be directed downwards and the parcel will be accelerated back towards where it came from.

The left-hand distribution (potential temperature decreases with height) is *statically unstable* in that any vertical displacement of lumps of air are augmented and magnified by the buoyancy forces. The right-hand distribution (potential temperature increases with height) *is statically stable*, because any vertical displacement is opposed and damped out by the buoyancy forces, while the middle case (potential temperature constant with height) is said to be *neutrally statically stable*.

Corresponding Conditions on Temperature

We do not measure potential temperature directly, so it is useful to write the above criteria in terms of temperature. We only need to consider the neutral case as this is the boundary of the other cases.

$$\frac{dT}{dz} = \frac{d}{dz}\left(\theta\left(\frac{p}{p_0}\right)^{\kappa}\right) = \theta\frac{d}{dz}\left(\frac{p}{p_0}\right)^{\kappa} = \theta p_0^{-\kappa}\kappa p^{\kappa-1}\frac{dp}{dz} = \frac{\kappa T}{p}(-\rho g)$$

where we have used in the appropriate places:- (1) the fact that $\frac{d\theta}{dz} = 0$ for this case and (2) the hydrostatic equation. If we use the gas law and the definition of θ we obtain for this neutral case:-

$$\frac{\mathrm{dT}}{\mathrm{dz}} = -\frac{\mathrm{g}}{\mathrm{C_p}}$$

Thus in the neutral case the temperature increases with height at a rate $-g/C_p$, i.e. temperature *decreases* with height at a rate $+g/C_p$. This is closely equal to 10K/km. If the temperature falls with height at a greater rate than this, we have the unstable case, while if it falls with height more slowly than g/C_p, we have the stable case. The situation is sketched in the Figure 2.

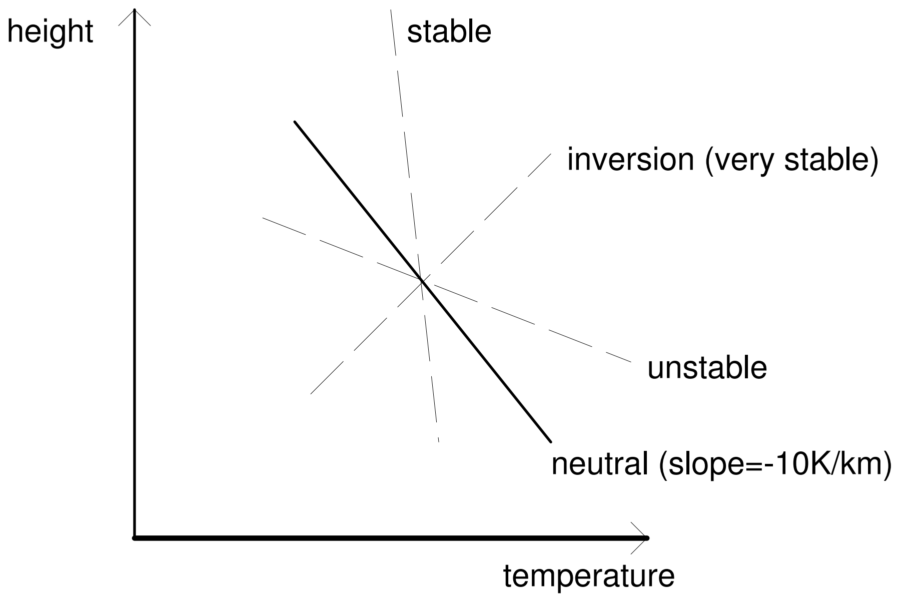


Figure 2 :Stable and unstable atmospheres

In the troposphere the temperature usually decreases with height but less strongly than 10K/km, so that it is usually stable.

Occasionally limited height ranges can be found in the troposphere where the temperature increases with height. Such a region is called an *inversion* because the temperature gradient is ``inverted" from the usual situation. Inversions are regions of high static stability.

Buoyancy Frequency

We have seen that in stable stratifications a raised parcel of air will accelerate downwards, and a depressed parcel of air will accelerate upwards. We can easily get an estimate of how large the accelerations are and hence how rapidly the status quo will be restored. Consider again Figure 1. Suppose A is at height z_0 and B (and A') at $z_0 + \delta z$. Let us suppose that parcel has a mass m', and that its volume when it reaches B is V. If its density is ρ' and the density of the surrounding fluid at B is ρ, then $m' = \rho' V$, while the mass of fluid which it has displaced is ρV. Thus, the nett upward force on the parcel = upthrust - gravitational force = $mg - m'g = V(\rho - \rho')g$. (The upthrust was calculated from Archimedes' principle). Thus, the upward acceleration of the parcel is this nett force divided by the mass, so that

$$\frac{d^2(\delta z)}{dt^2} = \frac{(\rho - \rho')}{\rho'} g$$

Now if the pressure of the parcel is the same as that of its surroundings we shall have

$$\frac{(\rho - \rho')}{\rho'} \approx \frac{(T' - T)}{T} = \frac{(\theta' - \theta)}{\theta},$$

the approximation being correct to first order. Now

$$\theta = \theta(z_0 + \delta z) = \theta(z_0) + \frac{d\theta}{dz}\delta z = \theta' + \frac{d\theta}{dz}\delta z$$

Thus to first order,

$$\frac{(\rho - \rho')}{\rho'} = -\frac{1}{\theta}\frac{d\theta}{dz}\delta z$$

and we have

$$\frac{d^2(\delta z)}{dt^2} = -\frac{g}{\theta}\frac{d\theta}{dz}\delta z$$

$\frac{g}{\theta}\frac{d\theta}{dz}$ is usually denoted by N^2, giving

$$\left(\frac{d}{dt}\right)^2 \delta z = -N^2 \delta z$$

The general form of the solutions to this equation is

$$\delta z = A\exp(\pm iNt)$$

where A is a constant.

For real N, this is a simple harmonic motion, with *angular* frequency[1] N and period, τ, given by

$$\tau = \frac{2\pi}{N}$$

N is called the Brunt-Vaisala[2] (angular) frequency after a British and a Swedish scientist who independently introduced this analysis.

This confirms our previous qualitative argument that when the potential temperature increases with height, air parcels which are somehow given an upward push will return to their original position, but it also takes that argument further, suggesting that it bobs up and down about its starting position and it indicates how rapidly the oscillation takes place. Of course we have drastically simplified what will really happen, as we have not taken into account that we have to move the environment about to let the chosen air parcel move up and down. This means that the potential energy which drives the motion is turned into kinetic energy of the parcel plus environment rather than of the parcel alone and the real frequency will be less than the Brunt-Vaisala frequency.

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[1] Note that we have carefully stated *angular frequency*. Frequency in the more usual sense of number of cycles per unit time would be the inverse of the period, namely $N/2\pi$.

[2] Each "a" in this name should strictly speaking have an umlaut (two dots) over it.